In[598]:=

```
(*This notebook is an ancillary file and
 contains supplemental material for the article
 "The Dirac equation as a linear
    tensor equation for one component".
  Long lines of output are truncated in
 the .pdf version of this notebook,
but the reader can view the notebook using
  the free-of-charge Wolfram Player.
   The notebook consists of 10 fragments,
which should be evaluated in succession.
  Evaluation of each of them takes no more
 than 2 minutes on the author's desktop.
  Mathematica version is 11.3.0.0. Short
 comments are provided at the end of
  the output of each fragment.*)
a = {{a11, a12}, {a21, a22}}; MatrixForm[a]
"a"
ah = ConjugateTranspose[a]; MatrixForm[ah]
"ah"
z2 = {{0, 0}, {0, 0}}; MatrixForm[z2]
"z2"
s1 = {{0, 1}, {1, 0}}; MatrixForm[s1]
s2 = {{0, -I}, {I, 0}}; MatrixForm[s2]
s3 = {{1, 0}, {0, -1}}; MatrixForm[s3]
"s1-3"
```

```
iun2 = {{1, 0}, {0, 1}}; MatrixForm[iun2]
"iun2"
taut = Array[z2 &, 4];
taut[[1]] = -iun2;
taut[[2]] = s1;
taut[[3]] = s2;
taut[[4]] = s3;
rhot = Array[z2 &, 4];
rhot[[1]] = -iun2;
rhot[[2]] = -s1;
rhot[[3]] = -s2;
rhot[[4]] = -s3;
taub = Array[z2 &, 4];
taub[[1]] = -iun2;
taub[[2]] = -s1;
taub[[3]] = -s2;
taub[[4]] = -s3;
lor = Array[0 &, {4, 4}];
For[i = 1, i ≤ 4, i++,
 For[j = 1, j ≤ 4, j++, lor[[i, j]] = FullSimplify[
     (1 / 2) Tr[rhot[[i]].a.taub[[j]].ah]]];];
MatrixForm[lor]
"lor"
mat1 =
 {{a11, a12, a21, a22}, {a21, a22, a11, a12},
```

```
  {-I a21, -I a22, I a11, I a12},
  {a11, a12, -a21, -a22}}; MatrixForm[mat1]
"mat1"
mat2 = {{Conjugate[a11], Conjugate[a12],
    -I Conjugate[a12], Conjugate[a11]},
  {Conjugate[a12], Conjugate[a11],
    I Conjugate[a11], -Conjugate[a12]},
  {Conjugate[a21], Conjugate[a22],
    -I Conjugate[a22], Conjugate[a21]},
  {Conjugate[a22], Conjugate[a21],
    I Conjugate[a21], -Conjugate[a22]}};
MatrixForm[mat2]
"mat2"
mat3 = mat1.mat2 / 2; MatrixForm[mat3]
"mat3"
FullSimplify[mat3 - lor]
g0 = Join[Join[z2, -iun2, 2],
  Join[-iun2, z2, 2]]; MatrixForm[g0]
g1 = Join[Join[z2, s1, 2], Join[-s1, z2, 2]];
MatrixForm[g1]
g2 = Join[Join[z2, s2, 2], Join[-s2, z2, 2]];
MatrixForm[g2]
g3 = Join[Join[z2, s3, 2], Join[-s3, z2, 2]];
MatrixForm[g3]
g5 = Join[Join[iun2, z2, 2],
```

Join[z2, -iun2, 2]]; MatrixForm[g5]

"g0-1-2-3-5"

s1, s2, s3 – Pauli matrices; taut[[1]], taut[[2]], taut[[3]], taut[[4]] – matrices $\tau^0, \tau^1, \tau^2, \tau^3$ from [17] (τ^0 is that from [17] times -1 because of the different choice of γ^0 in [17]); rhot[[1]], rhot[[2]], rhot[[3]], rhot[[4]] – matrices $\rho^0, \rho^1, \rho^2, \rho^3$ from [17] (ρ^0 is that from [17] times -1 because of the different choice of γ^0 in [17]); taub[[1]], taub[[2]], taub[[3]], taub[[4]] – matrices $\tau_0, \tau_1, \tau_2, \tau_3$ from [17] (τ^0 is that from [17] times -1 because of the different choice of γ^0 in [17]); a – matrix s in the following matrix transforming the spinor:

$$\begin{pmatrix} s & \\ & (s^\dagger)^{-1} \end{pmatrix}.$$

lor – the Lorentz transformation matrix $\Lambda = (\Lambda^\mu{}_\nu)$ corresponding to matrix s. lor is calculated using equation (60) of [17] with modified ρ_0 and τ_0. mat3 is calculated using equation (62) of [17] with modification due to the choice of γ_0. lor and mat3 coincide.

Out[598]//MatrixForm=

$$\begin{pmatrix} a11 & a12 \\ a21 & a22 \end{pmatrix}$$

Out[599]= a

Out[600]//MatrixForm=

$$\begin{pmatrix} \text{Conjugate}[a11] & \text{Conjugate}[a21] \\ \text{Conjugate}[a12] & \text{Conjugate}[a22] \end{pmatrix}$$

Out[601]= ah

Out[602]//MatrixForm=

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Out[603]= z2

Out[604]//MatrixForm=

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Out[605]//MatrixForm=

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Out[606]//MatrixForm=

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Out[607]= s1-3

Out[608]//MatrixForm=

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Out[609]= iun2

Out[626]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2}\left(\text{Abs}[a11]^2 + \text{Abs}[a12]^2 + \text{Abs}[\ldots \right. \\ \frac{1}{2}\,(a21\,\text{Conjugate}[a11] + a22\,\text{Conjugate}[a12] + a11\,\text{C}\ldots \\ \frac{1}{2}\,i\,(-a21\,\text{Conjugate}[a11] - a22\,\text{Conjugate}[a12] + a11\ldots \\ \frac{1}{2}\left(\text{Abs}[a11]^2 + \text{Abs}[a12]^2 - a21\,\text{Conjugate}[\ldots \right. \end{pmatrix}$$

Out[627]= lor

Out[628]//MatrixForm=

$$\begin{pmatrix} a11 & a12 & a21 & a22 \\ a21 & a22 & a11 & a12 \\ -i\,a21 & -i\,a22 & i\,a11 & i\,a12 \\ a11 & a12 & -a21 & -a22 \end{pmatrix}$$

Out[629]= mat1

Out[630]//MatrixForm=

$$\begin{pmatrix} \text{Conjugate}[a11] & \text{Conjugate}[a12] & -i\,\text{Conjugate}[a12] \\ \text{Conjugate}[a12] & \text{Conjugate}[a11] & i\,\text{Conjugate}[a11] \\ \text{Conjugate}[a21] & \text{Conjugate}[a22] & -i\,\text{Conjugate}[a22] \\ \text{Conjugate}[a22] & \text{Conjugate}[a21] & i\,\text{Conjugate}[a21] \end{pmatrix}$$

Out[631]= mat2

Out[632]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2}\,(a11\,\text{Conjugate}[a11] + a12\,\text{Conjugate}[a12] + a2\ldots \\ \frac{1}{2}\,(a21\,\text{Conjugate}[a11] + a22\,\text{Conjugate}[a12] + a1\ldots \\ \frac{1}{2}\,(-i\,a21\,\text{Conjugate}[a11] - i\,a22\,\text{Conjugate}[a12] + i\ldots \\ \frac{1}{2}\,(a11\,\text{Conjugate}[a11] + a12\,\text{Conjugate}[a12] - a2\ldots \end{pmatrix}$$

Out[633]= mat3

Out[634]= {{0, 0, 0, 0}, {0, 0, 0, 0},
{0, 0, 0, 0}, {0, 0, 0, 0}}

Out[635]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Out[636]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

Out[637]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

Out[638]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Out[639]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Out[640]= g0-1-2-3-5

Out[641]= s1, s2, s3 – Pauli matrices; taut[[1]], taut[[2]], taut[[3]], taut[[4]] – matrices $\tau^0, \tau^1, \tau^2, \tau^3$ from [17] (τ^0 is that from [17] times -1 because of the different choice of γ^0 in [17]); rhot[[1]], rhot[[2]], rhot[[3]], rhot[[4]] – matrices $\rho^0, \rho^1, \rho^2, \rho^3$ from [17] (ρ^0 is that from [17] times -1 because of the different choice of γ^0 in [17]); taub[[1]], taub[[2]], taub[[3]], taub[[4]] – matrices $\tau_0, \tau_1, \tau_2, \tau_3$ from [17] (τ^0 is that from [17] times -1 because of the different choice of γ^0 in [17]); a – matrix s in the following matrix transforming the spinor:

$$\begin{pmatrix} s & \\ & (s^\dagger)^{-1} \end{pmatrix}.$$

lor – the Lorentz transformation matrix $\Lambda = (\Lambda^\mu{}_\nu)$ corresponding to matrix s. lor is calculated using equation (60) of [17] with modified ρ_0 and τ_0. mat3 is calculated using equation (62) of [17] with modification due to the choice of γ_0. lor and mat3 coincide.

In[642]:=
```
ch = Join[Join[-I s2, z2, 2],
  Join[z2, I s2, 2]]; MatrixForm[ch]
"ch"
sig01 = I (g0.g1 - g1.g0) / 2; MatrixForm[sig01]
"sig01"
sig02 = I (g0.g2 - g2.g0) / 2; MatrixForm[sig02]
"sig02"
sig03 = I (g0.g3 - g3.g0) / 2; MatrixForm[sig03]
"sig03"
sig12 = I (g1.g2 - g2.g1) / 2; MatrixForm[sig12]
"sig12"
sig13 = I (g1.g3 - g3.g1) / 2; MatrixForm[sig13]
"sig13"
sig23 = I (g2.g3 - g3.g2) / 2; MatrixForm[sig23]
"sig23"
chi = {{chi1}, {chi2}, {chi3}, {chi4}};
MatrixForm[chi]
```

```
"chi"
zet = {{zet1}, {zet2}, {zet3}, {zet4}};
MatrixForm[zet]
"zet"
ch = Join[Join[-I s2, z2, 2],
  Join[z2, I s2, 2]]; MatrixForm[ch]
"ch"
spfi = {{0, 0, 0, 0},
  {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0,
 0}}; MatrixForm[spfi]
chit = Transpose[chi]; MatrixForm[chit]
"chit"
chitch = chit.ch; MatrixForm[chitch]
"chitch"
chit = Transpose[chi]; MatrixForm[chit]
"chit"
chih = Conjugate[chit]; MatrixForm[chih]
"chih"
chidj = chih.g0; MatrixForm[chidj]
"chidj"
zett = Transpose[zet]; MatrixForm[zett]
"zett"
zeth = Conjugate[zett]; MatrixForm[zeth]
"zeth"
zetdj = zeth.g0; MatrixForm[zetdj]
```

```
"zetdj"
zetch = ch.Transpose[zetdj]; MatrixForm[zetch]
"zetch"
spfi[[1, 2]] = FullSimplify[
   Flatten[chidj.sig01.zetch][[1]]];
spfi[[1, 3]] = FullSimplify[
   Flatten[chidj.sig02.zetch][[1]]];
spfi[[1, 4]] = FullSimplify[
   Flatten[chidj.sig03.zetch][[1]]];
spfi[[2, 3]] = FullSimplify[
   Flatten[chidj.sig12.zetch][[1]]];
spfi[[2, 4]] = FullSimplify[
   Flatten[chidj.sig13.zetch][[1]]];
spfi[[3, 4]] = FullSimplify[
   Flatten[chidj.sig23.zetch][[1]]];
spfi[[2, 1]] = -spfi[[1, 2]];
spfi[[3, 1]] = -spfi[[1, 3]];
spfi[[4, 1]] = -spfi[[1, 4]];
spfi[[3, 2]] = -spfi[[2, 3]];
spfi[[4, 2]] = -spfi[[2, 4]];
spfi[[4, 3]] = -spfi[[3, 4]];
MatrixForm[spfi]
"spfi"
spfitop =
 spfi /. {chi3 → 0, chi4 → 0, zet3 → 0,
```

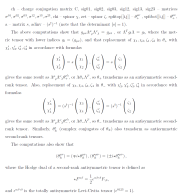
```
  zet4 → 0}; MatrixForm[spfitop]
  "spfitop"
spfibot =
  spfi /. {chi1 → 0, chi2 → 0, zet1 → 0,
  zet2 → 0}; MatrixForm[spfibot]
  "spfibot"
a = {{a11, a12}, {a21, a22}}; MatrixForm[a]
"a"
(*alpha→a11,beta→a12,gamma→a21,delta→a22*)
adinv =
  FullSimplify[Inverse[Transpose[Conjugate[a]]]]
"adinv"
FullSimplify[Transpose[Conjugate[a]].adinv /.
   a22 → (1 + a12 a21) / a1]
"Transpose[Conjugate[a]].adinv/.a22→(1+a12
   a21)/a1"
gg = {{1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, -1, 0},
   {0, 0, 0, -1}}; MatrixForm[gg]
"gg"
mat4 = FullSimplify[Transpose[mat3].gg.mat3];
MatrixForm[mat4]
"mat4"
MatrixForm[
  FullSimplify[mat4 /. {a22 → (1 + a21 a12) / a11}]]
spfi5top = FullSimplify[
```

```
  spfitop /. {zet1 → a11 zet1 + a12 zet2, zet2 →
      a21 zet1 + a22 zet2, chi1 → a11 chi1 + a12 chi2,
     chi2 → a21 chi1 + a22 chi2}];
MatrixForm[spfi5top]
"spfi5top"
spfi6top =
 FullSimplify[mat3.spfitop.Transpose[mat3]];
MatrixForm[spfi6top]
"spfi6top"
MatrixForm[FullSimplify[spfi5top - spfi6top /.
    {a22 → (1 + a21 a12) / a11}]]
"spfi5top-spfi6top"
spfi6ctop = FullSimplify[
  mat3.Conjugate[spfitop].Transpose[mat3]];
MatrixForm[spfi6ctop]
"spfi6ctop"
spfi5ctop = FullSimplify[Conjugate[spfitop] /.
    {zet1 → a11 zet1 + a12 zet2, zet2 →
      a21 zet1 + a22 zet2, chi1 → a11 chi1 + a12 chi2,
     chi2 → a21 chi1 + a22 chi2}];
MatrixForm[spfi5ctop]
"spfi5ctop"
MatrixForm[FullSimplify[spfi5ctop - spfi6ctop /.
    {a22 → (1 + a21 a12) / a11}]]
"spfi5ctop-spfi6ctop"
```

```
spfi5bot =
 FullSimplify[Conjugate[spfibot] /. {zet3 →
     Conjugate[a22] zet3 - Conjugate[a21] zet4,
    zet4 → -Conjugate[a12] zet3 +
      Conjugate[a11] zet4, chi3 →
     Conjugate[a22] chi3 - Conjugate[a21] chi4,
    chi4 → -Conjugate[a12] chi3 +
      Conjugate[a11] chi4}];
MatrixForm[spfi5bot]
"spfi5bot"
spfi6bot = FullSimplify[
  mat3.Conjugate[spfibot].Transpose[mat3]];
MatrixForm[spfi6bot]
"spfi6bot"
MatrixForm[FullSimplify[spfi5bot - spfi6bot /.
   {a22 → (1 + a21 a12) / a11}]]
"spfi5bot-spfi6bot"
spfi5ncbot = FullSimplify[
  spfibot /. {zet3 → Conjugate[a22] zet3 -
      Conjugate[a21] zet4, zet4 →
      -Conjugate[a12] zet3 + Conjugate[a11] zet4,
    chi3 → Conjugate[a22] chi3 - Conjugate[a21]
       chi4, chi4 → -Conjugate[a12] chi3 +
      Conjugate[a11] chi4}];
MatrixForm[spfi5ncbot]
```

```
"spfi5ncbot"
spfi6ncbot =
 FullSimplify[mat3.spfibot.Transpose[mat3]];
MatrixForm[spfi6ncbot]
"spfi6ncbot"
MatrixForm[
 FullSimplify[spfi5ncbot - spfi6ncbot /.
   {a22 → (1 + a21 a12) / a11}]]
"spfi5ncbot-spfi6ncbot"
MatrixForm[spfitop]
spfitopd =
 FullSimplify[(1 / 2) Activate[TensorContract[
     Inactive[TensorProduct][spfitop,
      LeviCivitaTensor[4], gg, gg],
     {{1, 7}, {2, 9}, {5, 8}, {6, 10}}]]];
MatrixForm[spfitopd]
"spfitopd"
FullSimplify[MatrixForm[spfitop + I spfitopd]]
"FullSimplify[MatrixForm[spfitop+I spfitopd]]"
FullSimplify[MatrixForm[Conjugate[spfitop] -
   I (1 / 2) Activate[TensorContract[
      Inactive[TensorProduct][Conjugate[
        spfitop], LeviCivitaTensor[4], gg, gg],
      {{1, 7}, {2, 9}, {5, 8}, {6, 10}}]]]]
"FullSimplify[MatrixForm[spfitop^*-I
```

```
  spfitopd^*]]"
(*FullSimplify[MatrixForm[
   Conjugate[spfibot]+I (1/2) Activate[
      TensorContract[Inactive[TensorProduct][
        Conjugate[spfibot],LeviCivitaTensor[4],
        gg, gg],{{1,7},{2,9},{5,8},{6,10}}]]]]
 "FullSimplify[MatrixForm[spfibot^*+I
   spfibotd^*]]"*)
spfibotd = FullSimplify[(1 / 2) Activate[
    TensorContract[Inactive[TensorProduct][
      spfibot, LeviCivitaTensor[4], gg, gg],
     {{1, 7}, {2, 9}, {5, 8}, {6, 10}}]]];
MatrixForm[spfibotd]
"spfibotd"
FullSimplify[MatrixForm[spfibot - I spfibotd]]
"FullSimplify[MatrixForm[spfibot-I spfibotd]]"
FullSimplify[MatrixForm[Conjugate[spfibot] +
   I (1 / 2) Activate[TensorContract[
      Inactive[TensorProduct][Conjugate[
        spfibot], LeviCivitaTensor[4], gg, gg],
      {{1, 7}, {2, 9}, {5, 8}, {6, 10}}]]]]
"FullSimplify[MatrixForm[spfibot^*+I
  spfibotd^*]]"
```



ch – charge conjugation matrix C, sig01, sig02, sig03, sig12, sig13, sig23 – matrices $\sigma^{01}, \sigma^{02}, \sigma^{03}, \sigma^{12}, \sigma^{13}, \sigma^{23}$, chi – spinor χ, zet – spinor ζ, spfitop[[i,j]] – $\theta_+^{\mu\nu}$, spfibot[[i,j]] – $\theta_+^{\mu\nu}$, a – matrix s, adinv – $(s^\dagger)^{-1}$ (note that the determinant $|s| = 1$).

The above computations show that $g_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\lambda = g_{\rho\lambda}$, or $\Lambda^T g_l \Lambda = g_l$, where the metric tensor with lower indices $g_l = (g_{\mu\nu})$, and that replacement of $\chi_1, \chi_2, \zeta_1, \zeta_2$ in θ_+ with $\chi_1', \chi_2', \zeta_1', \zeta_2'$ in accordance with formulas

$$\begin{pmatrix}\chi_1'\\ \chi_2'\end{pmatrix} = s\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}, \begin{pmatrix}\zeta_1'\\ \zeta_2'\end{pmatrix} = s\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix}$$

gives the same result as $\Lambda^\mu{}_\rho\Lambda^\nu{}_\lambda\theta_+^{\rho\lambda}$, or $\Lambda\theta_+\Lambda^T$, so θ_+ transforms as an antisymmetric second-rank tensor. Also, replacement of $\chi_3, \chi_4, \zeta_3, \zeta_4$ in θ_- with $\chi_3', \chi_4', \zeta_3', \zeta_4'$ in accordance with formulas

$$\begin{pmatrix}\chi_3'\\ \chi_4'\end{pmatrix} = (s^\dagger)^{-1}\begin{pmatrix}\chi_3\\ \chi_4\end{pmatrix}, \begin{pmatrix}\zeta_3'\\ \zeta_4'\end{pmatrix} = (s^\dagger)^{-1}\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix}$$

gives the same result as $\Lambda^\mu{}_\rho\Lambda^\nu{}_\lambda\theta_-^{\rho\lambda}$, or $\Lambda\theta_-\Lambda^T$, so θ_- transforms as an antisymmetric second-rank tensor. Similarly, $\theta_\pm^*$ (complex conjugates of $\theta_\pm$) also transform as antisymmetric second-rank tensors.

The computations also show that

$$(\theta_\pm^{\mu\nu}) = (\mp i \star\theta_\pm^{\mu\nu}), (\theta_\pm^{*\mu\nu}) = (\pm i \star\theta_\pm^{*\mu\nu}),$$

where the Hodge dual of a second-rank antisymmetric tensor is defined as

$$\star F^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta},$$

and $\epsilon^{\alpha\beta\gamma\delta}$ is the totally antisymmetric Levi-Civita tensor ($\epsilon^{0123} = 1$).

Out[642]//MatrixForm=

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Out[643]= ch

Out[644]//MatrixForm=

$$\begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix}$$

Out[645]= sig01

Out[646]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Out[647]= sig02

Out[648]//MatrixForm=

$$\begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}$$

Out[649]= sig03

Out[650]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Out[651]= sig12

Out[652]//MatrixForm=

$$\begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}$$

Out[653]= sig13

Out[654]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Out[655]= sig23

Out[656]//MatrixForm=

$$\begin{pmatrix} \text{chi1} \\ \text{chi2} \\ \text{chi3} \\ \text{chi4} \end{pmatrix}$$

Out[657]= chi

Out[658]//MatrixForm=

$$\begin{pmatrix} \text{zet1} \\ \text{zet2} \\ \text{zet3} \\ \text{zet4} \end{pmatrix}$$

Out[659]= zet

Out[660]//MatrixForm=

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Out[661]= ch

Out[662]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[663]//MatrixForm=

(chi1 chi2 chi3 chi4)

Out[664]= chit

Out[665]//MatrixForm=

(chi2 -chi1 -chi4 chi3)

Out[666]= chitch

Out[667]//MatrixForm=

(chi1 chi2 chi3 chi4)

Out[668]= chit

Out[669]//MatrixForm=

(Conjugate[chi1] Conjugate[chi2] Conjugate[chi3]

Out[670]= chih

Out[671]//MatrixForm=

(-Conjugate[chi3] -Conjugate[chi4] -Conjugate[c

Out[672]= chidj

Out[673]//MatrixForm=

(zet1 zet2 zet3 zet4)

Out[674]= zett

Out[675]//MatrixForm=

(Conjugate[zet1] Conjugate[zet2] Conjugate[zet3]

Out[676]= zeth

Out[677]//MatrixForm=

(-Conjugate[zet3] -Conjugate[zet4] -Conjugate[z

Out[678]= zetdj

Out[679]//MatrixForm=

```
( Conjugate[zet4]  )
( -Conjugate[zet3] )
( -Conjugate[zet2] )
( Conjugate[zet1]  )
```

Out[680]= zetch

Out[692]//MatrixForm=

```
(
 -ⅈ (Conjugate[chi1] Conjugate[zet1] - Conjugate[c
   -Conjugate[chi1] Conjugate[zet1] - Conjugate[ch
  ⅈ (Conjugate[chi2] Conjugate[zet1] + Conjugate[ch
```

Out[693]= spfi

Out[694]//MatrixForm=

```
(                                                  0
 -ⅈ (Conjugate[chi1] Conjugate[zet1] - Conjugate[c
   -Conjugate[chi1] Conjugate[zet1] - Conjugate[ch
  ⅈ (Conjugate[chi2] Conjugate[zet1] + Conjugate[ch
```

Out[695]= spfitop

Out[696]//MatrixForm=

```
(                                                  0
 -ⅈ (Conjugate[chi3] Conjugate[zet3] - Conjugate[c
   -Conjugate[chi3] Conjugate[zet3] - Conjugate[ch
  ⅈ (Conjugate[chi4] Conjugate[zet3] + Conjugate[ch
```

Out[697]= spfibot

Out[698]//MatrixForm=

$$\begin{pmatrix} a11 & a12 \\ a21 & a22 \end{pmatrix}$$

Out[699]= a

Out[700]=
```
{{Conjugate[a22] /
    (-Conjugate[a12] Conjugate[a21] +
      Conjugate[a11] Conjugate[a22]), Conjugate[
     a21] / (Conjugate[a12] Conjugate[a21] -
      Conjugate[a11] Conjugate[a22])},
  {Conjugate[a12] /
    (Conjugate[a12] Conjugate[a21] -
      Conjugate[a11] Conjugate[a22]), Conjugate[
     a11] / (-Conjugate[a12] Conjugate[a21] +
      Conjugate[a11] Conjugate[a22])}}
```

Out[701]= adinv

Out[702]= {{1, 0}, {0, 1}}

Out[703]=
```
Transpose[Conjugate[a]].adinv/.a22→(1+a12
  a21)/a1
```

Out[704]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Out[705]= gg

Out[706]//MatrixForm=

```
( (a12 a21 - a11 a22) (Conjugate[a12] Conjugate[a21]
                                                  0
                                                  0
                                                  0
```

Out[707]= mat4

Out[708]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Out[709]//MatrixForm=

```
( -ⅈ (Conjugate[a11 chi1 + a12 chi2] Conjugate[a11 ze
    -Conjugate[a11 chi1 + a12 chi2] Conjugate[a11 zet
  ⅈ (Conjugate[a21 chi1 + a22 chi2] Conjugate[a11 zet
```

Out[710]= spfi5top

Out[711]//MatrixForm=

```
( ⅈ (a12 a21 - a11 a22) (Conjugate[a11]^2 Conjugate[ch
   (a12 a21 - a11 a22) (Conjugate[a11]^2 Conjugate[chi
                                   -ⅈ (a12 a21 - a11 a2
```

Out[712]= spfi6top

Out[713]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[714]= spfi5top-spfi6top

Out[715]//MatrixForm=

$$\begin{pmatrix} \\ -\mathbb{i}\ \big(a11^2\ chi1\ zet1 - a21^2\ chi1\ zet1 + (a12 - a22)\ (a12 \\ \big(a11^2\ chi1\ zet1 + a21\ (a21\ chi1 + a22\ chi2)\ zet1 \\ \mathbb{i}\ (a11\ (2\ a21\ chi1\ zet1 + a22\ chi2\ zet1 + \end{pmatrix}$$

Out[716]= spfi6ctop

Out[717]//MatrixForm=

$$\begin{pmatrix} 0 \\ \mathbb{i}\ ((a11\ chi1 + a12\ chi2)\ (a11\ zet1 + a12\ zet2) - (a21 \\ -(a11\ chi1 + a12\ chi2)\ (a11\ zet1 + a12\ zet2) - (a21 \\ -\mathbb{i}\ ((a21\ chi1 + a22\ chi2)\ (a11\ zet1 + a12\ zet2) + (a1 \end{pmatrix}$$

Out[718]= spfi5ctop

Out[719]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[720]= spfi5ctop−spfi6ctop

Out[721]//MatrixForm=

$$\begin{pmatrix} \\ \mathbb{i}\ (-(chi4\ Conjugate[a11] - chi3\ Conjugate[a12])\ (z \\ -(chi4\ Conjugate[a11] - chi3\ Conjugate[a12])\ (ze \\ -\mathbb{i}\ (Conjugate[a12]\ ((chi4\ zet3 + chi3\ zet4)\ Con \end{pmatrix}$$

Out[722]= spfi5bot

Out[723]//MatrixForm=

```
⎛
  ⅈ (a12 a21 - a11 a22) (chi4 zet4 Conjugate[a11]^2 - (c
   (a12 a21 - a11 a22) (chi4 zet4 Conjugate[a11]^2 - (ch
  -ⅈ (a12 a21 - a11 a22) (2 chi4 zet4 Conjugate[a11] C
⎝
```

Out[724]= spfi6bot

Out[725]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[726]= spfi5bot-spfi6bot

Out[727]//MatrixForm=

```
⎛
  -ⅈ (-(a12 Conjugate[chi3] - a11 Conjugate[chi4]) (
      -(a12 Conjugate[chi3] - a11 Conjugate[chi4]) (a
              ⅈ (Conjugate[chi4] ((a12 a21 + a11 a22) Con
⎝
```

Out[728]= spfi5ncbot

Out[729]//MatrixForm=

```
⎛
  -ⅈ (Conjugate[a12] Conjugate[a21] - Conjugate[a11
               (Conjugate[a12] Conjugate[a21] - Conju
   ⅈ (Conjugate[a12] Conjugate[a21] - Conjugate[a11
⎝
```

Out[730]= spfi6ncbot

Out[731]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[732]= spfi5ncbot−spfi6ncbot

Out[733]//MatrixForm=

```
                                                0
-ⅈ (Conjugate[chi1] Conjugate[zet1] - Conjugate[c
  -Conjugate[chi1] Conjugate[zet1] - Conjugate[ch
 ⅈ (Conjugate[chi2] Conjugate[zet1] + Conjugate[ch
```

Out[734]//MatrixForm=

```
                                                0
   Conjugate[chi1] Conjugate[zet1] - Conjugate[chi
-ⅈ (Conjugate[chi1] Conjugate[zet1] + Conjugate[c
  -Conjugate[chi2] Conjugate[zet1] - Conjugate[ch
```

Out[735]= spfitopd

Out[736]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[737]= FullSimplify[MatrixForm[spfitop+I spfitopd]]

Out[738]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[739]= FullSimplify[MatrixForm[spfitop^*-I spfitopd^*]]

Out[740]//MatrixForm=

```
0
-Conjugate[chi3] Conjugate[zet3] + Conjugate[chi
i (Conjugate[chi3] Conjugate[zet3] + Conjugate[ch
Conjugate[chi4] Conjugate[zet3] + Conjugate[chi
```

Out[741]= spfibotd

Out[742]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[743]= FullSimplify[MatrixForm[spfibot-I spfibotd]]

Out[744]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[745]= FullSimplify[MatrixForm[spfibot^*+I spfibotd^*]]

ch – charge conjugation matrix C, sig01, sig02, sig03, sig12, sig13, sig23 – matrices $\sigma^{01}, \sigma^{02}, \sigma^{03}, \sigma^{12}, \sigma^{13}, \sigma^{23}$, chi – spinor χ, zet – spinor ζ, spfitop[[i,j]] – $\theta_+^{\mu\nu}$, spfibot[[i,j]] – $\theta_+^{\mu\nu}$, a – matrix s, adinv – $(s^\dagger)^{-1}$ (note that the determinant $|s| = 1$).

The above computations show that $g_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\lambda = g_{\rho\lambda}$, or $\Lambda^T g_l \Lambda = g_l$, where the metric tensor with lower indices $g_l = (g_{\mu\nu})$, and that replacement of $\chi_1, \chi_2, \zeta_1, \zeta_2$ in θ_+ with $\chi_1', \chi_2', \zeta_1', \zeta_2'$ in accordance with formulas

$$\begin{pmatrix} \chi_1' \\ \chi_2' \end{pmatrix} = s \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \begin{pmatrix} \zeta_1' \\ \zeta_2' \end{pmatrix} = s \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}$$

gives the same result as $\Lambda^\mu{}_\rho\Lambda^\nu{}_\lambda\theta_+^{\rho\lambda}$, or $\Lambda\theta_+\Lambda^T$, so θ_+ transforms as an antisymmetric second-rank tensor. Also, replacement of $\chi_3, \chi_4, \zeta_3, \zeta_4$ in θ_- with $\chi_3', \chi_4', \zeta_3', \zeta_4'$ in accordance with formulas

Out[746]=

$$\begin{pmatrix} \chi_3' \\ \chi_4' \end{pmatrix} = (s^\dagger)^{-1} \begin{pmatrix} \chi_3 \\ \chi_4 \end{pmatrix}, \begin{pmatrix} \zeta_3' \\ \zeta_4' \end{pmatrix} = (s^\dagger)^{-1} \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}$$

gives the same result as $\Lambda^\mu{}_\rho\Lambda^\nu{}_\lambda\theta_-^{\rho\lambda}$, or $\Lambda\theta_-\Lambda^T$, so θ_- transforms as an antisymmetric second-rank tensor. Similarly, $\theta_\pm^*$ (complex conjugates of $\theta_\pm$) also transform as antisymmetric second-rank tensors.

The computations also show that

$$(\theta_\pm^{\mu\nu}) = (\mp i\star\theta_\pm^{\mu\nu}), (\theta_\pm^{*\mu\nu}) = (\pm i\star\theta_\pm^{*\mu\nu}),$$

where the Hodge dual of a second-rank antisymmetric tensor is defined as

$$\star F^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta},$$

and $\epsilon^{\alpha\beta\gamma\delta}$ is the totally antisymmetric Levi-Civita tensor ($\epsilon^{0123} = 1$).

In[747]:=

```
psi = {{psi1}, {psi2}, {psi3}, {psi4}};
MatrixForm[psi]
"psi"
psit = Transpose[psi]; MatrixForm[psit]
"psit"
psich =
 ch.Transpose[ConjugateTranspose[psi].g0];
```

```
MatrixForm[psich]
"psich"
psichch =
 ch.Transpose[ConjugateTranspose[psich].g0];
MatrixForm[psichch]
"psichch"
psi - psichch
"psi-psichch"
FullSimplify[ConjugateTranspose[psich].g0 -
  Transpose[psi].ch]
"ConjugateTranspose[psich].g0-Transpose[psi].ch"

spfip = {{0, 0, 0, 0}, {0, 0, 0, 0},
  {0, 0, 0, 0}, {0, 0, 0,
 0}}; MatrixForm[spfip]
"spfip"
psitch = psit.ch; MatrixForm[psitch]
"psitch"
spfip[[1, 2]] =
  FullSimplify[Flatten[psitch.sig01.psi][[1]]];
spfip[[1, 3]] = FullSimplify[
   Flatten[psitch.sig02.psi][[1]]];
spfip[[1, 4]] = FullSimplify[
   Flatten[psitch.sig03.psi][[1]]];
spfip[[2, 3]] = FullSimplify[
```

```
   Flatten[psitch.sig12.psi][[1]]];
spfip[[2, 4]] = FullSimplify[
   Flatten[psitch.sig13.psi][[1]]];
spfip[[3, 4]] = FullSimplify[
   Flatten[psitch.sig23.psi][[1]]];
spfip[[2, 1]] = -spfip[[1, 2]];
spfip[[3, 1]] = -spfip[[1, 3]];
spfip[[4, 1]] = -spfip[[1, 4]];
spfip[[3, 2]] = -spfip[[2, 3]];
spfip[[4, 2]] = -spfip[[2, 4]];
spfip[[4, 3]] = -spfip[[3, 4]];
MatrixForm[spfip]
"spfip"
spfipbot = spfip /. {psi1 → 0, psi2 → 0};
MatrixForm[spfipbot]
 "spfipbot"
spfiptop = spfip /. {psi3 → 0, psi4 → 0};
MatrixForm[spfiptop]
 "spfiptop"
MatrixForm[chi]
"chi"
MatrixForm[chidj]
"chidj"
chichcon = ch.Transpose[chidj];
MatrixForm[chichcon]
```

```
"chichcon"
MatrixForm[psi]
"psi"
psidj = Conjugate[psit].g0; MatrixForm[psidj]
"psidj"
psichcon = ch.Transpose[psidj];
MatrixForm[psichcon]
"psichcon"
spfitop; MatrixForm[spfitop]
"spfitop"
spfibot; MatrixForm[spfibot]
"spfibot"
spfiptop; MatrixForm[spfiptop]
"spfiptop"
spfipbot; MatrixForm[spfipbot]
"spfipbot"
spfitopmod =
 spfitop /. {chi1 → psichcon[[1, 1]], chi2 →
    psichcon[[2, 1]], zet1 → psichcon[[1, 1]],
   zet2 → psichcon[[2, 1]]};
MatrixForm[spfitopmod]
"spfitopmod"
MatrixForm[FullSimplify[spfitopmod - spfipbot]]
"spfitopmod-spfipbot"
spfibotmod =
```

```
spfibot /. {chi3 → psichcon[[3, 1]], chi4 →
    psichcon[[4, 1]], zet3 → psichcon[[3, 1]],
   zet4 → psichcon[[4, 1]]};
MatrixForm[spfibotmod]
"spfibotmod"
MatrixForm[FullSimplify[spfibotmod - spfiptop]]
"spfibotmod-spfiptop"
FullSimplify[Activate[
  TensorContract[Inactive[TensorProduct][
    spfiptop, gg, gg, spfiptop],
   {{1, 4}, {2, 6}, {3, 7}, {5, 8}}]]]
"Activate[TensorContract[Inactive[TensorProduct]
  [spfiptop,gg,gg,spfiptop],{{1,4},{2,6},{3,7},
  {5,8}}]]"
FullSimplify[Activate[
  TensorContract[Inactive[TensorProduct][
    spfipbot, gg, gg, spfipbot],
   {{1, 4}, {2, 6}, {3, 7}, {5, 8}}]]]
"Activate[TensorContract[Inactive[TensorProduct]
  [spfipbot,gg,gg,spfipbot],{{1,4},{2,6},{3,7},
  {5,8}}]]"
```

psi – ψ, spfiptop[[i,j]] – $\psi_{+}^{\mu\nu}$, spfipbot[[i,j]] – $\psi_{-}^{\mu\nu}$. The computations show that $(\psi^c)^c = \psi$ and $\overline{\psi^c} = \psi^T C$. $(\psi_{\pm}^{\mu\nu})$ coinside with $(\theta_{\mp}^{\mu\nu})$ if $\chi = \psi^c$ and $\zeta = \psi^c$. $\psi_{\pm}^{\mu\nu}\psi_{\pm\mu\nu} = 0$.

Out[747]//MatrixForm=

$$\begin{pmatrix} \text{psi1} \\ \text{psi2} \\ \text{psi3} \\ \text{psi4} \end{pmatrix}$$

Out[748]= psi

Out[749]//MatrixForm=

$$\begin{pmatrix} \text{psi1} & \text{psi2} & \text{psi3} & \text{psi4} \end{pmatrix}$$

Out[750]= psit

Out[751]//MatrixForm=

$$\begin{pmatrix} \text{Conjugate[psi4]} \\ -\text{Conjugate[psi3]} \\ -\text{Conjugate[psi2]} \\ \text{Conjugate[psi1]} \end{pmatrix}$$

Out[752]= psich

Out[753]//MatrixForm=

$$\begin{pmatrix} \text{psi1} \\ \text{psi2} \\ \text{psi3} \\ \text{psi4} \end{pmatrix}$$

Out[754]= psichch

Out[755]= {{0}, {0}, {0}, {0}}

Out[756]= psi-psichch

Out[757]= {{0, 0, 0, 0}}

Out[758]= ConjugateTranspose[psich].g0-Transpose[psi].ch

Out[759]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[760]= spfip

Out[761]//MatrixForm=

$$\begin{pmatrix} \text{psi2} & -\text{psi1} & -\text{psi4} & \text{psi3} \end{pmatrix}$$

Out[762]= psitch

Out[774]//MatrixForm=

$$\begin{pmatrix} 0 & -i\left(\text{psi1}^2-\text{psi2}^2+\text{p}\right. \\ i\left(\text{psi1}^2-\text{psi2}^2+\text{psi3}^2-\text{psi4}^2\right) & 0 \\ -\text{psi1}^2-\text{psi2}^2-\text{psi3}^2-\text{psi4}^2 & -2\,\text{psi1}\,\text{psi2}+2 \\ -2\,i\,(\text{psi1}\,\text{psi2}+\text{psi3}\,\text{psi4}) & -i\left(\text{psi1}^2+\text{psi2}^2-\text{p}\right. \end{pmatrix}$$

Out[775]= spfip

Out[776]//MatrixForm=

$$\begin{pmatrix} 0 & -i\left(\text{psi3}^2-\text{psi4}^2\right) & \text{psi3}^2+\text{psi4}^2 \\ i\left(\text{psi3}^2-\text{psi4}^2\right) & 0 & -2\,\text{psi3}\,\text{psi4} \\ -\text{psi3}^2-\text{psi4}^2 & 2\,\text{psi3}\,\text{psi4} & 0 \\ -2\,i\,\text{psi3}\,\text{psi4} & -i\left(-\text{psi3}^2-\text{psi4}^2\right) & -\text{psi3}^2+\text{psi4}^2 \end{pmatrix}$$

Out[777]= spfipbot

Out[778]//MatrixForm=

$$\begin{pmatrix} 0 & -i\left(\text{psi1}^2-\text{psi2}^2\right) & \text{psi1}^2+\text{psi2}^2 & \\ i\left(\text{psi1}^2-\text{psi2}^2\right) & 0 & 2\,\text{psi1}\,\text{psi2} & i \\ -\text{psi1}^2-\text{psi2}^2 & -2\,\text{psi1}\,\text{psi2} & 0 & \\ -2\,i\,\text{psi1}\,\text{psi2} & -i\left(\text{psi1}^2+\text{psi2}^2\right) & \text{psi1}^2-\text{psi2}^2 & \end{pmatrix}$$

Out[779]= spfiptop

Out[780]//MatrixForm=

$$\begin{pmatrix} \text{chi1} \\ \text{chi2} \\ \text{chi3} \\ \text{chi4} \end{pmatrix}$$

Out[781]= chi

Out[782]//MatrixForm=

(-Conjugate[chi3] -Conjugate[chi4] -Conjugate[c

Out[783]= chidj

Out[784]//MatrixForm=

$$\begin{pmatrix} \text{Conjugate[chi4]} \\ -\text{Conjugate[chi3]} \\ -\text{Conjugate[chi2]} \\ \text{Conjugate[chi1]} \end{pmatrix}$$

Out[785]= chichcon

Out[786]//MatrixForm=

$$\begin{pmatrix} \text{psi1} \\ \text{psi2} \\ \text{psi3} \\ \text{psi4} \end{pmatrix}$$

Out[787]= psi

Out[788]//MatrixForm=

(-Conjugate[psi3] -Conjugate[psi4] -Conjugate[p

Out[789]= psidj

Out[790]//MatrixForm=

$$\begin{pmatrix} \text{Conjugate[psi4]} \\ -\text{Conjugate[psi3]} \\ -\text{Conjugate[psi2]} \\ \text{Conjugate[psi1]} \end{pmatrix}$$

Out[791]= psichcon

Out[792]//MatrixForm=

$$\begin{pmatrix} 0 \\ -i\,(\text{Conjugate}[\text{chi1}]\ \text{Conjugate}[\text{zet1}] - \text{Conjugate}[\text{c} \\ -\text{Conjugate}[\text{chi1}]\ \text{Conjugate}[\text{zet1}] - \text{Conjugate}[\text{ch} \\ i\,(\text{Conjugate}[\text{chi2}]\ \text{Conjugate}[\text{zet1}] + \text{Conjugate}[\text{ch} \end{pmatrix}$$

Out[793]= spfitop

Out[794]//MatrixForm=

$$\begin{pmatrix} 0 \\ -i\,(\text{Conjugate}[\text{chi3}]\ \text{Conjugate}[\text{zet3}] - \text{Conjugate}[\text{c} \\ -\text{Conjugate}[\text{chi3}]\ \text{Conjugate}[\text{zet3}] - \text{Conjugate}[\text{ch} \\ i\,(\text{Conjugate}[\text{chi4}]\ \text{Conjugate}[\text{zet3}] + \text{Conjugate}[\text{ch} \end{pmatrix}$$

Out[795]= spfibot

Out[796]//MatrixForm=

$$\begin{pmatrix} 0 & -i\left(\text{psi1}^2 - \text{psi2}^2\right) & \text{psi1}^2 + \text{psi2}^2 \\ i\left(\text{psi1}^2 - \text{psi2}^2\right) & 0 & 2\,\text{psi1}\,\text{psi2} & i \\ -\text{psi1}^2 - \text{psi2}^2 & -2\,\text{psi1}\,\text{psi2} & 0 \\ -2\,i\,\text{psi1}\,\text{psi2} & -i\left(\text{psi1}^2 + \text{psi2}^2\right) & \text{psi1}^2 - \text{psi2}^2 \end{pmatrix}$$

Out[797]= spfiptop

Out[798]//MatrixForm=

$$\begin{pmatrix} 0 & -i\left(\text{psi3}^2 - \text{psi4}^2\right) & \text{psi3}^2 + \text{psi4}^2 \\ i\left(\text{psi3}^2 - \text{psi4}^2\right) & 0 & -2\,\text{psi3}\,\text{psi4} \\ -\text{psi3}^2 - \text{psi4}^2 & 2\,\text{psi3}\,\text{psi4} & 0 \\ -2\,i\,\text{psi3}\,\text{psi4} & -i\left(-\text{psi3}^2 - \text{psi4}^2\right) & -\text{psi3}^2 + \text{psi4}^2 \end{pmatrix}$$

Out[799]= spfipbot

Out[800]//MatrixForm=

$$\begin{pmatrix} 0 & i\left(-psi3^2+psi4^2\right) & psi3^2+psi4 \\ -i\left(-psi3^2+psi4^2\right) & 0 & -2\,psi3\,psi4 \\ -psi3^2-psi4^2 & 2\,psi3\,psi4 & 0 \\ -2\,i\,psi3\,psi4 & i\left(psi3^2+psi4^2\right) & -psi3^2+psi4 \end{pmatrix}$$

Out[801]= spfitopmod

Out[802]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[803]= spfitopmod−spfipbot

Out[804]//MatrixForm=

$$\begin{pmatrix} 0 & i\left(-psi1^2+psi2^2\right) & psi1^2+psi2^2 \\ -i\left(-psi1^2+psi2^2\right) & 0 & 2\,psi1\,psi2 \\ -psi1^2-psi2^2 & -2\,psi1\,psi2 & 0 \\ -2\,i\,psi1\,psi2 & i\left(-psi1^2-psi2^2\right) & psi1^2-psi2^2 \end{pmatrix}$$

Out[805]= spfibotmod

Out[806]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[807]= spfibotmod−spfiptop

Out[808]= 0

Out[809]=
```
Activate[TensorContract[Inactive[TensorProduct][
  spfiptop,gg,gg,spfiptop],{{1,4},{2,6},{3,7},{
  5,8}}]]
```

Out[810]= 0

Out[811]=
```
Activate[TensorContract[Inactive[TensorProduct][
  spfipbot,gg,gg,spfipbot],{{1,4},{2,6},{3,7},{
  5,8}}]]
```

Out[812]= psi – ψ, spfiptop[[i,j]] – $\psi_+^{\mu\nu}$, spfipbot[[i,j]] – $\psi_-^{\mu\nu}$. The computations show that $(\psi^c)^c = \psi$ and $\overline{\psi^c} = \psi^T C$, $(\psi_\pm^{\mu\nu})$ coinside with $(\theta_\mp^{\mu\nu})$ if $\chi = \psi^c$ and $\zeta = \psi^c$, $\psi_\pm^{\mu\nu}\psi_{\pm\mu\nu} = 0$.

In[813]:=
```
ksi = {{ksi1}, {ksi2}, {ksi3}, {ksi4}};
MatrixForm[ksi]
"ksi"
ksit = Transpose[ksi]; MatrixForm[ksit]
"ksit"
ksih = Conjugate[ksit]; MatrixForm[ksih]
"ksih"
ksidj = ksih.g0; MatrixForm[ksidj]
"ksidj"
ksich = ch.Transpose[ksidj]; MatrixForm[ksich]
"ksich"
etach = ksich /.
  {ksi1 → eta1, ksi2 → eta2, ksi3 → eta3,
 ksi4 → eta4}; MatrixForm[etach]
"etach"
etadj = ksidj /.
```

```
  {ksi1 → eta1, ksi2 → eta2, ksi3 → eta3,
 ksi4 → eta4}
"etadj"
ksichtop = FullSimplify[ksich /.
   {ksi3 → 0, ksi4 → 0, eta3 → 0,  eta4 → 0}];
MatrixForm[ksichtop]
ksidjtop = FullSimplify[ksidj /.
   {ksi3 → 0, ksi4 → 0, eta3 → 0,  eta4 → 0}];
MatrixForm[ksidjtop]
etachtop = FullSimplify[etach /.
   {ksi3 → 0, ksi4 → 0, eta3 → 0,  eta4 → 0}];
MatrixForm[etachtop]
etadjtop = FullSimplify[etadj /.
   {ksi3 → 0, ksi4 → 0, eta3 → 0,  eta4 → 0}];
MatrixForm[etadjtop]
"ksi ch dj eta ch dj top"
ksidjtop.etachtop
"ksidjtop.etachtop"
attop = spfitop /.
  {chi1 -> ksi1, chi2 -> ksi2, chi3 -> ksi3,
   chi4 -> ksi4, zet1 → ksi1, zet2 -> ksi2,
   zet3 -> ksi3, zet4 -> ksi4}; MatrixForm[attop]
"attop"
FullSimplify[Activate[TensorContract[
   Inactive[TensorProduct][attop, gg, gg, attop],
```

```
    {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]]
"attop^2"
bttop = spfitop /.
   {chi1 -> ksi1, chi2 -> ksi2, chi3 -> ksi3,
    chi4 -> ksi4, zet1 → eta1, zet2 -> eta2,
    zet3 -> eta3, zet4 -> eta4}; MatrixForm[bttop]
"bttop"
FullSimplify[Activate[TensorContract[
    Inactive[TensorProduct][bttop, gg, gg, bttop],
    {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]]
"bttop^2"
FullSimplify[Activate[TensorContract[
     Inactive[TensorProduct][bttop, gg, gg,
      bttop], {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]] /
  (ksidjtop.etachtop) ^2
"bttop^2/(ksidjtop.etachtop)^2"
cttop = spfitop /.
   {chi1 → eta1, chi2 -> eta2, chi3 -> eta3,
    chi4 -> eta4, zet1 → eta1, zet2 -> eta2,
    zet3 -> eta3, zet4 -> eta4}; MatrixForm[cttop]
"cttop"
FullSimplify[Activate[TensorContract[
    Inactive[TensorProduct][cttop, gg, gg, cttop],
    {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]]
"cttop^2"
```

```
FullSimplify[Activate[TensorContract[
   Inactive[TensorProduct][attop, gg, gg, bttop],
   {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]]
"attop bttop"
FullSimplify[Activate[TensorContract[
   Inactive[TensorProduct][attop, gg, gg, cttop],
   {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]]
"attop cttop"
FullSimplify[Activate[TensorContract[
    Inactive[TensorProduct][attop, gg, gg,
     cttop], {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]] /
 (ksidjtop.etachtop) ^2
"attop cttop/(ksidjtop.etachtop)^2"
FullSimplify[Activate[TensorContract[
   Inactive[TensorProduct][bttop, gg, gg, cttop],
   {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]]
"bttop cttop"
```

```
ksichbot = FullSimplify[ksich /.
   {ksi1 → 0, ksi2 → 0, eta1 → 0,  eta2 → 0}];
MatrixForm[ksichbot]
ksidjbot = FullSimplify[ksidj /.
   {ksi1 → 0, ksi2 → 0, eta1 → 0,  eta2 → 0}];
```

```
MatrixForm[ksidjbot]
etachbot = FullSimplify[etach /.
   {ksi1 → 0, ksi2 → 0, eta1 → 0,  eta2 → 0}];
MatrixForm[etachbot]
etadjbot = FullSimplify[etadj /.
   {ksi1 → 0, ksi2 → 0, eta1 → 0,  eta2 → 0}];
MatrixForm[etadjbot]
"ksi ch dj eta ch dj bot"
ksidjbot.etachbot
"ksidjbot.etachbot"
atbot = spfibot /.
  {chi1 -> ksi1, chi2 -> ksi2, chi3 -> ksi3,
   chi4 -> ksi4, zet1 → ksi1, zet2 -> ksi2,
   zet3 -> ksi3, zet4 -> ksi4}; MatrixForm[atbot]
"atbot"
FullSimplify[Activate[TensorContract[
   Inactive[TensorProduct][atbot, gg, gg, atbot],
   {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]]
"atbot^2"
btbot = spfibot /.
  {chi1 -> ksi1, chi2 -> ksi2, chi3 -> ksi3,
   chi4 -> ksi4, zet1 → eta1, zet2 -> eta2,
   zet3 -> eta3, zet4 -> eta4}; MatrixForm[btbot]
"btbot"
FullSimplify[Activate[TensorContract[
```

```
    Inactive[TensorProduct][btbot, gg, gg, btbot],
    {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]]
"btbot^2"
FullSimplify[Activate[TensorContract[
     Inactive[TensorProduct][btbot, gg, gg,
      btbot], {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]] /
   (ksidjbot.etachbot) ^2]
"btbot^2/(ksidjbot.etachbot)^2"
ctbot = spfibot /.
   {chi1 → eta1, chi2 -> eta2, chi3 -> eta3,
    chi4 -> eta4, zet1 → eta1, zet2 -> eta2,
    zet3 -> eta3, zet4 -> eta4}; MatrixForm[ctbot]
"ctbot"
FullSimplify[Activate[TensorContract[
    Inactive[TensorProduct][ctbot, gg, gg, ctbot],
    {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]]
"ctbot^2"
FullSimplify[Activate[TensorContract[
    Inactive[TensorProduct][atbot, gg, gg, btbot],
    {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]]
"atbot btbot"
FullSimplify[Activate[TensorContract[
    Inactive[TensorProduct][atbot, gg, gg, ctbot],
    {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]]
"atbot ctbot"
```

```
FullSimplify[Activate[TensorContract[
    Inactive[TensorProduct][atbot, gg, gg,
     ctbot], {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]] /
  (ksidjbot.etachbot)^2]
"atbot ctbot/(ksidjbot.etachbot)^2"
FullSimplify[Activate[TensorContract[
   Inactive[TensorProduct][btbot, gg, gg, ctbot],
   {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]]
"btbot ctbot"
```

ksi – ξ, eta – η, attop[[i,j]] or atbot[[i,j]] (depending on whether ξ and η are eigenvectors of γ^5 with eigenvalue +1 or −1) – $u^{\mu\nu}$, bttop[[i,j]] or btbot[[i,j]] – $v^{\mu\nu}$, cttop[[i,j]] or ctbot[[i,j]] – $w^{\mu\nu}$. The computations show that, if $\bar{\xi}\eta^c = 1$, then $u^{\mu\nu}w_{\mu\nu} = -8$, $v^{\mu\nu}w_{\mu\nu} = 0$, $u^{\mu\nu}w_{\mu\nu} = 0$ and $u^{\mu\nu}u_{\mu\nu} = 0$, $u^{\mu\nu}v_{\mu\nu} = 0$, $v^{\mu\nu}v_{\mu\nu} = 4$.

Out[813]//MatrixForm=

$$\begin{pmatrix} \text{ksi1} \\ \text{ksi2} \\ \text{ksi3} \\ \text{ksi4} \end{pmatrix}$$

Out[814]= ksi

Out[815]//MatrixForm=

(ksi1 ksi2 ksi3 ksi4)

Out[816]= ksit

Out[817]//MatrixForm=

(Conjugate[ksi1] Conjugate[ksi2] Conjugate[ksi3]

Out[818]= ksih

Out[819]//MatrixForm=

(-Conjugate[ksi3] -Conjugate[ksi4] -Conjugate[k

Out[820]= ksidj

Out[821]//MatrixForm=

$$\begin{pmatrix} \text{Conjugate[ksi4]} \\ -\text{Conjugate[ksi3]} \\ -\text{Conjugate[ksi2]} \\ \text{Conjugate[ksi1]} \end{pmatrix}$$

Out[822]= ksich

Out[823]//MatrixForm=

$$\begin{pmatrix} \text{Conjugate[eta4]} \\ -\text{Conjugate[eta3]} \\ -\text{Conjugate[eta2]} \\ \text{Conjugate[eta1]} \end{pmatrix}$$

Out[824]= etach

Out[825]= {{-Conjugate[eta3], -Conjugate[eta4], -Conjugate[eta1], -Conjugate[eta2]}}

Out[826]= etadj

Out[827]//MatrixForm=

$$\begin{pmatrix} 0 \\ 0 \\ -\text{Conjugate[ksi2]} \\ \text{Conjugate[ksi1]} \end{pmatrix}$$

Out[828]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & -\text{Conjugate[ksi1]} & -\text{Conjugate[ksi2]} \end{pmatrix}$$

Out[829]//MatrixForm=

$$\begin{pmatrix} 0 \\ 0 \\ -\text{Conjugate[eta2]} \\ \text{Conjugate[eta1]} \end{pmatrix}$$

Out[830]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & -\text{Conjugate[eta1]} & -\text{Conjugate[eta2]} \end{pmatrix}$$

Out[831]= ksi ch dj eta ch dj top

Out[832]= {{Conjugate[eta2] Conjugate[ksi1] -
Conjugate[eta1] Conjugate[ksi2]}}

Out[833]= ksidjtop.etachtop

Out[834]//MatrixForm=

0	ⅈ (Conju
-ⅈ (Conjugate[ksi1]^2 - Conjugate[ksi2]^2)	
-Conjugate[ksi1]^2 - Conjugate[ksi2]^2	-2 Con
2 ⅈ Conjugate[ksi1] Conjugate[ksi2]	ⅈ (Conju

Out[835]= attop

Out[836]= 0

Out[837]= attop^2

Out[838]//MatrixForm=

0
-ⅈ (Conjugate[eta1] Conjugate[ksi1] - Conjugate[e
-Conjugate[eta1] Conjugate[ksi1] - Conjugate[et
ⅈ (Conjugate[eta2] Conjugate[ksi1] + Conjugate[et

Out[839]= bttop

Out[840]= 4 (Conjugate[eta2] Conjugate[ksi1] -
Conjugate[eta1] Conjugate[ksi2])^2

Out[841]= bttop^2

Out[842]= {{4}}

Out[843]= bttop^2/(ksidjtop.etachtop)^2

Out[844]//MatrixForm=

$$\begin{pmatrix} 0 & i\,(\text{Conju} \\ -i\,(\text{Conjugate}[\text{eta1}]^2 - \text{Conjugate}[\text{eta2}]^2) & \\ -\text{Conjugate}[\text{eta1}]^2 - \text{Conjugate}[\text{eta2}]^2 & -2\,\text{Con} \\ 2\,i\,\text{Conjugate}[\text{eta1}]\,\text{Conjugate}[\text{eta2}] & i\,(\text{Conju} \end{pmatrix}$$

Out[845]= cttop

Out[846]= 0

Out[847]= cttop^2

Out[848]= 0

Out[849]= attop bttop

Out[850]= $-8\,(\text{Conjugate}[\text{eta2}]\,\text{Conjugate}[\text{ksi1}] - \text{Conjugate}[\text{eta1}]\,\text{Conjugate}[\text{ksi2}])^2$

Out[851]= attop cttop

Out[852]= {{-8}}

Out[853]= attop cttop / (ksidjtop.etachtop)^2

Out[854]= 0

Out[855]= bttop cttop

Out[856]//MatrixForm=

$$\begin{pmatrix} \text{Conjugate}[\text{ksi4}] \\ -\text{Conjugate}[\text{ksi3}] \\ 0 \\ 0 \end{pmatrix}$$

Out[857]//MatrixForm=

$$\begin{pmatrix} -\text{Conjugate}[\text{ksi3}] & -\text{Conjugate}[\text{ksi4}] & 0 & 0 \end{pmatrix}$$

Out[858]//MatrixForm=

$$\begin{pmatrix} \text{Conjugate[eta4]} \\ -\text{Conjugate[eta3]} \\ 0 \\ 0 \end{pmatrix}$$

Out[859]//MatrixForm=

$$\begin{pmatrix} -\text{Conjugate[eta3]} & -\text{Conjugate[eta4]} & 0 & 0 \end{pmatrix}$$

Out[860]= ksi ch dj eta ch dj bot

Out[861]= {{-Conjugate[eta4] Conjugate[ksi3] +
Conjugate[eta3] Conjugate[ksi4]}}

Out[862]= ksidjbot.etachbot

Out[863]//MatrixForm=

$$\begin{pmatrix} 0 & \text{i} \left(\text{Conj}\right. \\ -\text{i} \left(\text{Conjugate[ksi3]}^2 - \text{Conjugate[ksi4]}^2\right) & \\ -\text{Conjugate[ksi3]}^2 - \text{Conjugate[ksi4]}^2 & 2\ \text{Con} \\ 2\ \text{i}\ \text{Conjugate[ksi3] Conjugate[ksi4]} & \text{i} \left(-\text{Conj}\right. \end{pmatrix}$$

Out[864]= atbot

Out[865]= 0

Out[866]= atbot^2

Out[867]//MatrixForm=

$$\begin{pmatrix} 0 \\ -\text{i} \left(\text{Conjugate[eta3] Conjugate[ksi3]} - \text{Conjugate[e}\right. \\ -\text{Conjugate[eta3] Conjugate[ksi3]} - \text{Conjugate[et} \\ \text{i} \left(\text{Conjugate[eta4] Conjugate[ksi3]} + \text{Conjugate[et}\right. \end{pmatrix}$$

Out[868]= btbot

Out[869]= 4 (Conjugate[eta4] Conjugate[ksi3] -
Conjugate[eta3] Conjugate[ksi4])^2

Out[870]= btbot^2

Out[871]= {{4}}

Out[872]= btbot^2/(ksidjbot.etachbot)^2

Out[873]//MatrixForm=

```
0                                                          i (Conju
-i (Conjugate[eta3]^2 - Conjugate[eta4]^2)
-Conjugate[eta3]^2 - Conjugate[eta4]^2                     2 Con
2 i Conjugate[eta3] Conjugate[eta4]                        i (-Conj
```

Out[874]= ctbot

Out[875]= 0

Out[876]= ctbot^2

Out[877]= 0

Out[878]= atbot btbot

Out[879]= -8 (Conjugate[eta4] Conjugate[ksi3] - Conjugate[eta3] Conjugate[ksi4])^2

Out[880]= atbot ctbot

Out[881]= {{-8}}

Out[882]= atbot ctbot/(ksidjbot.etachbot)^2

Out[883]= 0

Out[884]= btbot ctbot

Out[885]= ksi – ξ, eta – η, attop[[i,j]] or atbot[[i,j]] (depending on whether ξ and η are eigenvectors of γ^5 with eigenvalue $+1$ or -1) – $u^{\mu\nu}$, bttop[[i,j]] or btbot[[i,j]] – $v^{\mu\nu}$, cttop[[i,j]] or ctbot[[i,j]] – $w^{\mu\nu}$. The computations show that, if $\bar{\xi}\eta^c = 1$, then $u^{\mu\nu}w_{\mu\nu} = -8, v^{\mu\nu}w_{\mu\nu} = 0, w^{\mu\nu}w_{\mu\nu} = 0$ and $u^{\mu\nu}u_{\mu\nu} = 0, u^{\mu\nu}v_{\mu\nu} = 0, v^{\mu\nu}v_{\mu\nu} = 4$.

In[886]:=
```
"top"
u0 = {{0, u01, u02, u03},
   {-u01, 0, 0, 0}, {-u02, 0, 0, 0},
   {-u03, 0, 0, 0}}; MatrixForm[u0]
"u0"
u = u0 - I FullSimplify[
     (1 / 2) Activate[TensorContract[Inactive[
          TensorProduct][u0, LeviCivitaTensor[4],
         gg, gg], {{1, 8}, {2, 10}, {3, 7},
         {4, 9}}]]]; MatrixForm[u]
"u"
FullSimplify[u + I FullSimplify[(1 / 2) Activate[
      TensorContract[Inactive[TensorProduct][
        u, LeviCivitaTensor[4], gg, gg],
       {{1, 8}, {2, 10}, {3, 7}, {4, 9}}]]]]

FullSimplify[Activate[TensorContract[
   Inactive[TensorProduct][u, gg, gg, u],
   {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]]
"u^2"

v0 = {{0, v01, v02, v03},
   {-v01, 0, 0, 0}, {-v02, 0, 0, 0},
   {-v03, 0, 0, 0}}; MatrixForm[v0]
"v0"
```

```
v = v0 - I FullSimplify[
    (1 / 2) Activate[TensorContract[Inactive[
        TensorProduct][v0, LeviCivitaTensor[4],
        gg, gg], {{1, 8}, {2, 10}, {3, 7},
        {4, 9}}]]]; MatrixForm[v]
"v"
FullSimplify[v + I FullSimplify[(1 / 2) Activate[
      TensorContract[Inactive[TensorProduct][
        v, LeviCivitaTensor[4], gg, gg],
       {{1, 8}, {2, 10}, {3, 7}, {4, 9}}]]]]

FullSimplify[Activate[TensorContract[
   Inactive[TensorProduct][v, gg, gg, v],
   {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]]
"v^2"
FullSimplify[Activate[TensorContract[
   Inactive[TensorProduct][u, gg, gg, v],
   {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]]
"u v"

umod = FullSimplify[
  Conjugate[u0 + I FullSimplify[(1 / 2) Activate[
        TensorContract[Inactive[TensorProduct][
          u0, LeviCivitaTensor[4], gg, gg],
         {{1, 8}, {2, 10}, {3, 7},
```

```
            {4, 9}}]]]]]; MatrixForm[umod]
"umod"
FullSimplify[Activate[TensorContract[
   Inactive[TensorProduct][umod, gg, gg, umod],
   {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]]
"umod^2"
FullSimplify[
 umod + I FullSimplify[(1 / 2) Activate[
      TensorContract[Inactive[TensorProduct][
        umod, LeviCivitaTensor[4], gg,  gg],
       {{1, 8}, {2, 10}, {3, 7}, {4, 9}}]]]]
uumod = FullSimplify[Activate[TensorContract[
    Inactive[TensorProduct][u, gg, gg, umod],
    {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]]
"u umod"
vumod = FullSimplify[Activate[TensorContract[
    Inactive[TensorProduct][v, gg, gg, umod],
    {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]]
"v umod"

al1 = FullSimplify[-vumod ^ 2 / uumod ^ 2]
"al1"
al2 = FullSimplify[2 vumod / uumod]
"al2"
al3 = FullSimplify[-8 / uumod]
```

```
"al3"
w = FullSimplify[al1 u + al2 v + al3 umod];
MatrixForm[w]
"w"
al2 4 +
 al3 (-4 (v01 Conjugate[u01] + v02 Conjugate[u02] +
      v03 Conjugate[u03]))
"v w"
al3 uumod
"u w"
FullSimplify[al2^2 4 + 2 al1 al3
    (-4 (Abs[u01]^2 + Abs[u02]^2 + Abs[u03]^2)) +
  2 al2 al3 (-4 (v01 Conjugate[u01] +
        v02 Conjugate[u02] + v03 Conjugate[u03]))]
"w^2"
(*FullSimplify[Activate[TensorContract[
     Inactive[TensorProduct][w,gg,gg,w],
     {{1,3},{2,5},{4,7},{6,8}}]]/.
   {u03→Sqrt[-u01^2-u02^2],v03→
     -(u01 v01+u02 v02)/Sqrt[-u01^2-u02^2]}]
 "w^2"*)
(*FullSimplify[Activate[TensorContract[
    Inactive[TensorProduct][u,gg,gg,w],
    {{1,3},{2,5},{4,7},{6,8}}]]]
 "u w"*)
```

```
"bot"
u0 = {{0, u01, u02, u03},
   {-u01, 0, 0, 0}, {-u02, 0, 0, 0},
   {-u03, 0, 0, 0}}; MatrixForm[u0]
"u0"
u = u0 + I FullSimplify[
     (1 / 2) Activate[TensorContract[Inactive[
          TensorProduct][u0, LeviCivitaTensor[4],
         gg, gg], {{1, 8}, {2, 10}, {3, 7},
         {4, 9}}]]]; MatrixForm[u]
"u"
FullSimplify[u - I FullSimplify[(1 / 2) Activate[
      TensorContract[Inactive[TensorProduct][
         u, LeviCivitaTensor[4], gg, gg],
        {{1, 8}, {2, 10}, {3, 7}, {4, 9}}]]]]

FullSimplify[Activate[TensorContract[
    Inactive[TensorProduct][u, gg, gg, u],
    {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]]
"u^2"

v0 = {{0, v01, v02, v03},
   {-v01, 0, 0, 0}, {-v02, 0, 0, 0},
   {-v03, 0, 0, 0}}; MatrixForm[v0]
"v0"
```

```
v = v0 + I FullSimplify[
    (1 / 2) Activate[TensorContract[Inactive[
        TensorProduct][v0, LeviCivitaTensor[4],
       gg, gg], {{1, 8}, {2, 10}, {3, 7},
       {4, 9}}]]]; MatrixForm[v]
"v"
FullSimplify[v - I FullSimplify[(1 / 2) Activate[
     TensorContract[Inactive[TensorProduct][
       v, LeviCivitaTensor[4], gg, gg],
      {{1, 8}, {2, 10}, {3, 7}, {4, 9}}]]]]

FullSimplify[Activate[TensorContract[
   Inactive[TensorProduct][v, gg, gg, v],
   {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]]
"v^2"
FullSimplify[Activate[TensorContract[
   Inactive[TensorProduct][u, gg, gg, v],
   {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]]
"u v"

umod = FullSimplify[
  Conjugate[u0 - I FullSimplify[(1 / 2) Activate[
       TensorContract[Inactive[TensorProduct][
         u0, LeviCivitaTensor[4], gg, gg],
        {{1, 8}, {2, 10}, {3, 7},
```

```
          {4, 9}}]]]]]; MatrixForm[umod]
"umod"
FullSimplify[
 umod - I FullSimplify[(1 / 2) Activate[
      TensorContract[Inactive[TensorProduct][
        umod, LeviCivitaTensor[4], gg, gg],
       {{1, 8}, {2, 10}, {3, 7}, {4, 9}}]]]]
uumod = FullSimplify[Activate[TensorContract[
    Inactive[TensorProduct][u, gg, gg, umod],
    {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]]
"u umod"
vumod = FullSimplify[Activate[TensorContract[
    Inactive[TensorProduct][v, gg, gg, umod],
    {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]]
"v umod"

al1 = FullSimplify[-vumod^2 / uumod^2]
"al1"
al2 = FullSimplify[2 vumod / uumod]
"al2"
al3 = FullSimplify[-8 / uumod]
"al3"
w = FullSimplify[al1 u + al2 v + al3 umod];
MatrixForm[w]
"w"
```

This fragment of computation contains two similar parts corresponding to the cases where ξ and η are eigenvectors of γ^5 with eigenvalue $+1$ and -1. The computation illustrates the more explicit solution for $w^{\mu\nu}$ (where a specific tensor $k^{\mu\nu}$ is chosen) written in coordinates.

u[[i,j]] – $u^{\mu\nu}$, v[[i,j]] – $v^{\mu\nu}$, umod[[i,j]] – $k^{\mu\nu}$, w[[i,j]] – $w^{\mu\nu}$, u01 – u_1, u02 – u_2, u03 – u_3, v01 – v_1, v02 – v_2, v03 – v_3, al1 – α_1, al2 – α_2, al3 – α_3.

The computations show that $u^{\mu\nu}u_{\mu\nu} = -4((u_1)^2+(u_2)^2+(u_3)^2) = 0$, $u^{\mu\nu}v_{\mu\nu} = -4(u_1v_1 + u_2v_2 + u_3v_3) = 0$, $v^{\mu\nu}v_{\mu\nu} = -4((v_1)^2+(v_2)^2+(v_3)^2) = 4$, $u^{\mu\nu}k_{\mu\nu} = -4(|u_1|^2+|u_2|^2+|u_3|^2)$, and $w^{\mu\nu}w_{\mu\nu} = -8$, $v^{\mu\nu}w_{\mu\nu} = 0$, $u^{\mu\nu}w_{\mu\nu} = 0$.

Out[886]= top

Out[887]//MatrixForm=

$$\begin{pmatrix} 0 & u01 & u02 & u03 \\ -u01 & 0 & 0 & 0 \\ -u02 & 0 & 0 & 0 \\ -u03 & 0 & 0 & 0 \end{pmatrix}$$

Out[888]= u0

Out[889]//MatrixForm=

$$\begin{pmatrix} 0 & u01 & u02 & u03 \\ -u01 & 0 & i\, u03 & -i\, u02 \\ -u02 & -i\, u03 & 0 & i\, u01 \\ -u03 & i\, u02 & -i\, u01 & 0 \end{pmatrix}$$

Out[890]= u

Out[891]= {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

Out[892]= $-4\left(u01^2 + u02^2 + u03^2\right)$

Out[893]= u^2

Out[894]//MatrixForm=

$$\begin{pmatrix} 0 & v01 & v02 & v03 \\ -v01 & 0 & 0 & 0 \\ -v02 & 0 & 0 & 0 \\ -v03 & 0 & 0 & 0 \end{pmatrix}$$

Out[895]= v0

Out[896]//MatrixForm=

$$\begin{pmatrix} 0 & v01 & v02 & v03 \\ -v01 & 0 & i\, v03 & -i\, v02 \\ -v02 & -i\, v03 & 0 & i\, v01 \\ -v03 & i\, v02 & -i\, v01 & 0 \end{pmatrix}$$

Out[897]= v

Out[898]= {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

Out[899]= $-4\left(v01^2 + v02^2 + v03^2\right)$

Out[900]= v^2

Out[901]= $-4\,(u01\,v01 + u02\,v02 + u03\,v03)$

Out[902]= u v

Out[903]//MatrixForm=

$$\begin{pmatrix} 0 & \text{Conjugate}[u01] & \text{Conjugate}[u\ldots \\ -\text{Conjugate}[u01] & 0 & i\,\text{Conjugate}[u\ldots \\ -\text{Conjugate}[u02] & -i\,\text{Conjugate}[u03] & 0 \\ -\text{Conjugate}[u03] & i\,\text{Conjugate}[u02] & -i\,\text{Conjugate}[\ldots \end{pmatrix}$$

Out[904]= umod

Out[905]= $-4\left(\text{Conjugate}[u01]^2 + \text{Conjugate}[u02]^2 + \text{Conjugate}[u03]^2\right)$

Out[906]= umod^2

Out[907]= {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

Out[908]= $-4\left(\text{Abs}[u01]^2 + \text{Abs}[u02]^2 + \text{Abs}[u03]^2\right)$

Out[909]= u umod

Out[910]= $-4\,(v01\,\text{Conjugate}[u01] + v02\,\text{Conjugate}[u02] + v03\,\text{Conjugate}[u03])$

Out[911]= v umod

Out[912]= $-\left((v01\,\text{Conjugate}[u01] + v02\,\text{Conjugate}[u02] + v03\,\text{Conjugate}[u03])^2 \Big/ \left(\text{Abs}[u01]^2 + \text{Abs}[u02]^2 + \text{Abs}[u03]^2\right)^2\right)$

Out[913]= al1

Out[914]= $(2\,(v01\,\text{Conjugate}[u01] + v02\,\text{Conjugate}[u02] + v03\,\text{Conjugate}[u03])) \Big/ \left(\text{Abs}[u01]^2 + \text{Abs}[u02]^2 + \text{Abs}[u03]^2\right)$

Out[915]= al2

Out[916]= $$\frac{2}{\text{Abs}[u01]^2 + \text{Abs}[u02]^2 + \text{Abs}[u03]^2}$$

Out[917]= al3

Out[918]//MatrixForm=

$-2\left(\text{Abs}[u01]^2+\text{Abs}[u02]^2+\text{Abs}[u03]^2\right)\,\text{Conjugate}[u01]-2\,v01\,\left(\text{Abs}[u01]^2+\text{Al}\right.$

$-2\left(\text{Abs}[u01]^2+\text{Abs}[u02]^2+\text{Abs}[u03]^2\right)\,\text{Conjugate}[u02]-2\,v02\,\left(\text{Abs}[u01]^2+\text{Al}\right.$

$-2\left(\text{Abs}[u01]^2+\text{Abs}[u02]^2+\text{Abs}[u03]^2\right)\,\text{Conjugate}[u03]-2\,v03\,\left(\text{Abs}[u01]^2+\text{Al}\right.$

Out[919]= w

Out[920]= 0

Out[921]= v w

Out[922]= -8

Out[923]= u w

Out[924]= 0

Out[925]= w^2

Out[926]= bot

Out[927]//MatrixForm=

$$\begin{pmatrix} 0 & u01 & u02 & u03 \\ -u01 & 0 & 0 & 0 \\ -u02 & 0 & 0 & 0 \\ -u03 & 0 & 0 & 0 \end{pmatrix}$$

Out[928]= u0

Out[929]//MatrixForm=

$$\begin{pmatrix} 0 & u01 & u02 & u03 \\ -u01 & 0 & -i\,u03 & i\,u02 \\ -u02 & i\,u03 & 0 & -i\,u01 \\ -u03 & -i\,u02 & i\,u01 & 0 \end{pmatrix}$$

Out[930]= u

Out[931]= {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

Out[932]= $-4\left(u01^2 + u02^2 + u03^2\right)$

Out[933]= u^2

Out[934]//MatrixForm=

$$\begin{pmatrix} 0 & v01 & v02 & v03 \\ -v01 & 0 & 0 & 0 \\ -v02 & 0 & 0 & 0 \\ -v03 & 0 & 0 & 0 \end{pmatrix}$$

Out[935]= v0

Out[936]//MatrixForm=

$$\begin{pmatrix} 0 & v01 & v02 & v03 \\ -v01 & 0 & -i\,v03 & i\,v02 \\ -v02 & i\,v03 & 0 & -i\,v01 \\ -v03 & -i\,v02 & i\,v01 & 0 \end{pmatrix}$$

Out[937]= v

Out[938]= {{0, 0, 0, 0}, {0, 0, 0, 0},
{0, 0, 0, 0}, {0, 0, 0, 0}}

Out[939]= $-4\left(v01^2 + v02^2 + v03^2\right)$

Out[940]= v^2

Out[941]= −4 (u01 v01 + u02 v02 + u03 v03)

Out[942]= u v

Out[943]//MatrixForm=

$$\begin{pmatrix} 0 & \text{Conjugate}[u01] & \text{Conjugate}[u0 \\ -\text{Conjugate}[u01] & 0 & -i\,\text{Conjugate}[\\ -\text{Conjugate}[u02] & i\,\text{Conjugate}[u03] & 0 \\ -\text{Conjugate}[u03] & -i\,\text{Conjugate}[u02] & i\,\text{Conjugate}[u \end{pmatrix}$$

Out[944]= umod

Out[945]= {{0, 0, 0, 0}, {0, 0, 0, 0},
{0, 0, 0, 0}, {0, 0, 0, 0}}

Out[946]= $-4\left(\text{Abs}[u01]^2 + \text{Abs}[u02]^2 + \text{Abs}[u03]^2\right)$

Out[947]= u umod

Out[948]= −4 (v01 Conjugate[u01] +
v02 Conjugate[u02] + v03 Conjugate[u03])

Out[949]= v umod

Out[950]= $-\left((v01\,\text{Conjugate}[u01] + v02\,\text{Conjugate}[u02] + v03\,\text{Conjugate}[u03])^2 \Big/ \left(\text{Abs}[u01]^2 + \text{Abs}[u02]^2 + \text{Abs}[u03]^2\right)^2\right)$

Out[951]= al1

Out[952]= $(2\,(v01\,\text{Conjugate}[u01] + v02\,\text{Conjugate}[u02] + v03\,\text{Conjugate}[u03])) \Big/ \left(\text{Abs}[u01]^2 + \text{Abs}[u02]^2 + \text{Abs}[u03]^2\right)$

Out[953]= al2

Out[954]= $$\frac{2}{\text{Abs}[u01]^2 + \text{Abs}[u02]^2 + \text{Abs}[u03]^2}$$

Out[955]= al3

Out[956]//MatrixForm=

$-2\left(\text{Abs}[u01]^2+\text{Abs}[u02]^2+\text{Abs}[u03]^2\right)\text{Conjugate}[u01]-2\,v01\left(\text{Abs}[u01]^2+\text{Al}\right.$

$-2\left(\text{Abs}[u01]^2+\text{Abs}[u02]^2+\text{Abs}[u03]^2\right)\text{Conjugate}[u02]-2\,v02\left(\text{Abs}[u01]^2+\text{Al}\right.$

$-2\left(\text{Abs}[u01]^2+\text{Abs}[u02]^2+\text{Abs}[u03]^2\right)\text{Conjugate}[u03]-2\,v03\left(\text{Abs}[u01]^2+\text{Al}\right.$

Out[957]= w

Out[958]=

This fragment of computation contains two similar parts corresponding to the cases where ξ and η are eigenvectors of γ^5 with eigenvalue $+1$ and -1. The computation illustrates the more explicit solution for $w^{\mu\nu}$ (where a specific tensor $k^{\mu\nu}$ is chosen) written in coordinates.

u[[i,j]] – $u^{\mu\nu}$, v[[i,j]] – $v^{\mu\nu}$, umod[[i,j]] – $k^{\mu\nu}$, w[[i,j]] – $w^{\mu\nu}$, u01 – u_1, u02 – u_2, u03 – u_3, v01 – v_1, v02 – v_2, v03 – v_3, al1 – α_1, al2 – α_2, al3 – α_3.

The computations show that $u^{\mu\nu}u_{\mu\nu} = -4((u_1)^2+(u_2)^2+(u_3)^2) = 0$, $u^{\mu\nu}v_{\mu\nu} = -4(u_1v_1 + u_2v_2 + u_3v_3) = 0$, $v^{\mu\nu}v_{\mu\nu} = -4((v_1)^2 + (v_2)^2 + (v_3)^2) = 4$, $u^{\mu\nu}k_{\mu\nu} = -4(|u_1|^2 + |u_2|^2 + |u_3|^2)$, and $u^{\mu\nu}w_{\mu\nu} = -8, v^{\mu\nu}w_{\mu\nu} = 0, w^{\mu\nu}w_{\mu\nu} = 0$.

In[959]:=
```
attopspfipbot =
  FullSimplify[Activate[TensorContract[Inactive[
        TensorProduct][attop, gg, gg, spfipbot],
     {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]]
"attopspfipbot"
ksidjpsitop = ksidj.psi /. {ksi3 → 0, ksi4 → 0}
"ksidjpsitop"
ksidjpsibot = ksidj.psi /. {ksi1 → 0, ksi2 → 0}
"ksidjpsibot"
atbotspfiptop =
  FullSimplify[Activate[TensorContract[Inactive[
        TensorProduct][atbot, gg, gg, spfiptop],
     {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]]
"atbotspfiptop"
jc0 = FullSimplify[(psidj.g0.psi)[[1, 1]]]
"jc0"
jc1 = FullSimplify[(psidj.g1.psi)[[1, 1]]]
"jc1"
jc2 = FullSimplify[(psidj.g2.psi)[[1, 1]]]
```

```
"jc2"
jc3 = FullSimplify[(psidj.g3.psi)[[1, 1]]]
"jc3"
jsqbot = FullSimplify[Activate[TensorContract[
    Inactive[TensorProduct][gg, spfipbot,
     Conjugate[spfipbot]], {{1, 3}, {2, 5}}]]];
MatrixForm[jsqbot]
"jsqbot"
jsqbot2 = FullSimplify[
  {{jc0 jc0, jc0 jc1, jc0 jc2, jc0 jc3},
    {jc1 jc0, jc1 jc1, jc1 jc2, jc1 jc3},
    {jc2 jc0, jc2 jc1, jc2 jc2, jc2 jc3},
    {jc3 jc0, jc3 jc1, jc3 jc2, jc3 jc3}} /.
   {psi1 → 0, psi2 → 0}]; MatrixForm[jsqbot2]
"jsqbot2"
FullSimplify[jsqbot + 2 jsqbot2]
"jsqbot+2 jsqbot2"
jsqtop = FullSimplify[Activate[TensorContract[
    Inactive[TensorProduct][gg, spfiptop,
     Conjugate[spfiptop]], {{1, 3}, {2, 5}}]]];
MatrixForm[jsqtop]
"jsqtop"
jsqtop2 = FullSimplify[
  {{jc0 jc0, jc0 jc1, jc0 jc2, jc0 jc3},
    {jc1 jc0, jc1 jc1, jc1 jc2, jc1 jc3},
```

```
    {jc2 jc0, jc2 jc1, jc2 jc2, jc2 jc3},
    {jc3 jc0, jc3 jc1, jc3 jc2, jc3 jc3}} /.
   {psi3 → 0, psi4 → 0}]; MatrixForm[jsqtop2]
"jsqtop2"
FullSimplify[jsqtop + 2 jsqtop2]
"jsqtop+2 jsqtop2"
```

ksidj – $\bar{\xi}$, psi – ψ, psidj – $\bar{\psi}$, jc0, jc1, jc2, jc3 – components of the current j^μ.

The computation shows that, if ξ is an eigenvector of γ^5 with eigenvalue +1, then $\bar{\xi}\psi = \bar{\xi}\psi_- = -\psi_3\xi_1^* - \psi_4\xi_2^*$, if ξ is an eigenvector of γ^5 with eigenvalue −1, then $\bar{\xi}\psi = \bar{\xi}\psi_+ = -\psi_1\xi_3^* - \psi_2\xi_4^*$. Also, the computation shows that $\psi^{\mu\nu}_{\mp}a_{\mu\nu} = -8(\bar{\xi}\psi)^2$ and $j^{\mu\nu}_{\pm} = g_{\rho\lambda}\psi^{\rho\mu}_{\pm}\left(\psi^{\lambda\nu}_{\pm}\right)^* = -2j^\mu_{\pm}j^\nu_{\pm}$.

Out[959]= -8 (psi3 Conjugate[ksi1] + psi4 Conjugate[ksi2])^2

Out[960]= attopspfipbot

Out[961]= {{-psi3 Conjugate[ksi1] - psi4 Conjugate[ksi2]}}

Out[962]= ksidjpsitop

Out[963]= {{-psi1 Conjugate[ksi3] - psi2 Conjugate[ksi4]}}

Out[964]= ksidjpsibot

Out[965]= -8 (psi1 Conjugate[ksi3] + psi2 Conjugate[ksi4])^2

Out[966]= atbotspfiptop

Out[967]= Abs[psi1]^2 + Abs[psi2]^2 + Abs[psi3]^2 + Abs[psi4]^2

Out[968]= jc0

Out[969]= psi2 Conjugate[psi1] + psi1 Conjugate[psi2] -
 psi4 Conjugate[psi3] - psi3 Conjugate[psi4]

Out[970]= jc1

Out[971]= i (-psi2 Conjugate[psi1] + psi1 Conjugate[psi2] +
 psi4 Conjugate[psi3] - psi3 Conjugate[psi4])

Out[972]= jc2

Out[973]= Abs[psi1]^2 + Abs[psi4]^2 - psi2 Conjugate[psi2] - psi3 Conjugate[psi3]

Out[974]= jc3

Out[975]//MatrixForm=

$$\begin{pmatrix} -2\ (\text{Abs}[\text{psi3}]^2 + \text{Abs}[\text{psi4}]^2 \\ 2\ (\text{Abs}[\text{psi3}]^2 + \text{Abs}[\text{psi4}]^2)\ (\text{psi4 Conjugate}[\text{psi3}] \\ -2\ i\ (\text{Abs}[\text{psi3}]^2 + \text{Abs}[\text{psi4}]^2)\ (\text{psi4 Conjugate}[\text{psi3} \\ 2\ (\text{Abs}[\text{psi3}]^4 - \text{Abs}[\text{psi4}]^4 \end{pmatrix}$$

Out[976]= jsqbot

Out[977]//MatrixForm=

$$\begin{pmatrix} (\text{Abs}[\text{psi3}]^2 + \text{Abs}[\text{psi4}]^2)^2 \\ -(\text{Abs}[\text{psi3}]^2 + \text{Abs}[\text{psi4}]^2)\ (\text{psi4 Conjugate}[\text{psi3}] + \\ i\ (\text{Abs}[\text{psi3}]^2 + \text{Abs}[\text{psi4}]^2)\ (\text{psi4 Conjugate}[\text{psi3}] - \\ -\text{Abs}[\text{psi3}]^4 + \text{Abs}[\text{psi4}]^4 \end{pmatrix}$$

Out[978]= jsqbot2

Out[979]= {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

Out[980]= jsqbot+2 jsqbot2

Out[981]//MatrixForm=

$$\begin{pmatrix} -2\ (\text{Abs}[\text{psi1}]^2 + \text{Abs}[\text{psi2}]^2) \\ -2\ (\text{Abs}[\text{psi1}]^2 + \text{Abs}[\text{psi2}]^2)\ (\text{psi2 Conjugate}[\text{psi1}] \\ 2\ i\ (\text{Abs}[\text{psi1}]^2 + \text{Abs}[\text{psi2}]^2)\ (\text{psi2 Conjugate}[\text{psi1}] \\ -2\ \text{Abs}[\text{psi1}]^4 + 2\ \text{Abs}[\text{psi2}]^4 \end{pmatrix}$$

Out[982]= jsqtop

Out[983]//MatrixForm=

```
(Abs[psi1]^2 + Abs[psi2]^2)^2
(Abs[psi1]^2 + Abs[psi2]^2) (psi2 Conjugate[psi1] +
i (Abs[psi1]^2 + Abs[psi2]^2) (-psi2 Conjugate[psi1]
Abs[psi1]^4 - Abs[psi2]^4
```

Out[984]= jsqtop2

Out[985]= {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

Out[986]= jsqtop+2 jsqtop2

ksidj – $\bar{\xi}$, psi – ψ, psidj – $\bar{\psi}$, jc0, jc1, jc2, jc3 – components of the current j^μ.

Out[987]= The computation shows that, if ξ is an eigenvector of γ^5 with eigenvalue +1, then $\bar{\xi}\psi = \bar{\xi}\psi_- = -\psi_3\xi_1^* - \psi_4\xi_2^*$, if ξ is an eigenvector of γ^5 with eigenvalue −1, then $\bar{\xi}\psi = \bar{\xi}\psi_+ = -\psi_1\xi_3^* - \psi_2\xi_4^*$. Also, the computation shows that $\psi_\mp^{\mu\nu}u_{\mu\nu} = -8(\bar{\xi}\psi)^2$ and $j_\pm^{\mu\nu} = g_{\sigma\lambda}\psi_\pm^{\sigma\mu}\left(\psi_\pm^{\lambda\nu}\right)^* = -2j_\pm^\mu j_\pm^\nu$.

In[988]:=

```
zer = Join[Join[z2, z2, 2], Join[z2, z2, 2]];
MatrixForm[zer]
"zer"
sig = Array[zer &, {4, 4}];
sig[[1, 2]] = sig01;
sig[[1, 3]] = sig02;
sig[[1, 4]] = sig03;
sig[[2, 3]] = sig12;
sig[[2, 4]] = sig13;
sig[[3, 4]] = sig23;
sig[[2, 1]] = -sig[[1, 2]];
sig[[3, 1]] = -sig[[1, 3]];
```

```
sig[[4, 1]] = -sig[[1, 4]];
sig[[3, 2]] = -sig[[2, 3]];
sig[[4, 2]] = -sig[[2, 4]];
sig[[4, 3]] = -sig[[3, 4]];
gmt = {g0, g1, g2, g3}
"gmt"
bb = {bb0, bb1, bb2, bb3};
cc = {cc0, cc1, cc2, cc3};
bbr = {bb0, -bb1, -bb2, -bb3};
ccr = {cc0, -cc1, -cc2, -cc3};
gsigg = Array[zer &, {4, 4, 4, 4}];
For[i = 1, i ≤ 4, i++, For[j = 1, j ≤ 4, j++,
   For[k = 1, k ≤ 4, k++, For[l = 1, l ≤ 4, l++,
       gsigg[[i, j, k, l]] = FullSimplify[
         gmt[[i]].sig[[j, k]].gmt[[l]]]];];];]
ksks2top = Array[0 &, {4, 4}];
For[i = 1, i ≤ 4, i++,
 For[j = 1, j ≤ 4, j++, For[k = 1, k ≤ 4, k++,
     For[l = 1, l ≤ 4, l++, ksks2top[[j, k]] =
        ksks2top[[j, k]] - bb[[i]] bb[[l]]
          ((ksidjtop.gsigg[[i, j, k, l]].
              ksichtop)[[1, 1]])];];];]
ksks2top = FullSimplify[ksks2top];
MatrixForm[ksks2top]
"ksks2top"
```

```
ksks3top =
  -2 FullSimplify[Activate[TensorContract[
       Inactive[TensorProduct][bb, gg, bb,
        attop], {{1, 2}, {4, 6}}]]] + 2 Transpose[
     FullSimplify[Activate[TensorContract[
        Inactive[TensorProduct][bb, gg,
         bb, attop], {{1, 2}, {4, 6}}]]]] -
   FullSimplify[Activate[TensorContract[
       Inactive[TensorProduct][bb, gg, bb],
       {{1, 2}, {3, 4}}]]] attop ;
MatrixForm[FullSimplify[ksks2top - ksks3top]]
"ksks2top-ksks3top"
kset2top = Array[0 &, {4, 4}];
For[i = 1, i ≤ 4, i++,
 For[j = 1, j ≤ 4, j++, For[k = 1, k ≤ 4, k++,
     For[l = 1, l ≤ 4, l++, kset2top[[j, k]] =
        kset2top[[j, k]] - bb[[i]] cc[[l]]
          ((ksidjtop.gsigg[[i, j, k, l]].
               etachtop)[[1, 1]])];];];]
kset2top = FullSimplify[kset2top];
MatrixForm[kset2top]
"kset2top"
etks2top = Array[0 &, {4, 4}];
For[i = 1, i ≤ 4, i++,
 For[j = 1, j ≤ 4, j++, For[k = 1, k ≤ 4, k++,
```

```
      For[l = 1, l ≤ 4, l++, etks2top[[j, k]] =
          etks2top[[j, k]] - cc[[i]] bb[[l]]
            ((etadjtop.gsigg[[i, j, k, l]].
                ksichtop)[[1, 1]])];];];]
etks2top = FullSimplify[etks2top];
MatrixForm[etks2top]
"etks2top"
bc27 = Array[0 &, {4, 4}];
For[j = 1, j ≤ 4, j++,
 For[k = 1, k ≤ 4, k++, bc27[[j, k]] =
      (-2 I bbr[[j]] ccr[[k]] + 2 I bbr[[k]] ccr[[
           j]]) (ksidjtop.etachtop)[[1, 1]];];]
bceps = 2 (ksidjtop.etachtop)[[1, 1]]
  FullSimplify[Activate[TensorContract[
      Inactive[TensorProduct][bb, cc,
       LeviCivitaTensor[4]], {{1, 3}, {2, 6}}]]];
MatrixForm[bceps]
"bceps"
kset3top =
  -2 FullSimplify[Activate[TensorContract[
        Inactive[TensorProduct][bb, gg, cc,
         bttop], {{1, 2}, {4, 6}}]]] + 2 Transpose[
      FullSimplify[Activate[TensorContract[
          Inactive[TensorProduct][bb, gg,
           cc, bttop], {{1, 2}, {4, 6}}]]]] +
```

```
    2 FullSimplify[Activate[TensorContract[
        Inactive[TensorProduct][bb, cc, gg,
         bttop], {{2, 3}, {1, 5}}]]] - 2 Transpose[
      FullSimplify[Activate[TensorContract[
         Inactive[TensorProduct][bb, cc,
          gg, bttop], {{2, 3}, {1, 5}}]]]] -
    2 FullSimplify[Activate[TensorContract[
        Inactive[TensorProduct][bb, cc, gg],
        {{1, 4}, {2, 3}}]]] bttop ;
MatrixForm[FullSimplify[kset2top +
    etks2top - bc27 - kset3top - bceps]]
"kset2top+etks2top-bc-kset3top-bceps"
etet2top = Array[0 &, {4, 4}];
For[i = 1, i ≤ 4, i++,
 For[j = 1, j ≤ 4, j++, For[k = 1, k ≤ 4, k++,
    For[l = 1, l ≤ 4, l++, etet2top[[j, k]] =
       etet2top[[j, k]] - cc[[i]] cc[[l]]
         ((etadjtop.gsigg[[i, j, k, l]].
             etachtop)[[1, 1]])];];];]
etet2top = FullSimplify[etet2top];
MatrixForm[etet2top]
"etet2top"
etet3top =
  -2 FullSimplify[Activate[TensorContract[
        Inactive[TensorProduct][cc, gg, cc,
```

```
        cttop], {{1, 2}, {4, 6}}]]] + 2 Transpose[
      FullSimplify[Activate[TensorContract[
        Inactive[TensorProduct][cc, gg,
         cc, cttop], {{1, 2}, {4, 6}}]]]] -
    FullSimplify[Activate[TensorContract[
       Inactive[TensorProduct][cc, gg, cc],
       {{1, 2}, {3, 4}}]]] cttop;
MatrixForm[FullSimplify[etet2top - etet3top]]
"etet2top-etet3top"
ksks2bot = Array[0 &, {4, 4}];
For[i = 1, i ≤ 4, i++,
 For[j = 1, j ≤ 4, j++, For[k = 1, k ≤ 4, k++,
     For[l = 1, l ≤ 4, l++, ksks2bot[[j, k]] =
        ksks2bot[[j, k]] - bb[[i]] bb[[l]]
          ((ksidjbot.gsigg[[i, j, k, l]].
              ksichbot)[[1, 1]])];];];]
ksks2bot = FullSimplify[ksks2bot];
MatrixForm[ksks2bot]
"ksks2bot"
ksks3bot =
  -2 FullSimplify[Activate[TensorContract[
       Inactive[TensorProduct][bb, gg, bb,
        atbot], {{1, 2}, {4, 6}}]]] + 2 Transpose[
      FullSimplify[Activate[TensorContract[
        Inactive[TensorProduct][bb, gg,
```

```
          bb, atbot], {{1, 2}, {4, 6}}]]]] -
    FullSimplify[Activate[TensorContract[
        Inactive[TensorProduct][bb, gg, bb],
        {{1, 2}, {3, 4}}]]] atbot ;
MatrixForm[FullSimplify[ksks2bot - ksks3bot]]
"ksks2bot-ksks3bot"
kset2bot = Array[0 &, {4, 4}];
For[i = 1, i ≤ 4, i++,
 For[j = 1, j ≤ 4, j++, For[k = 1, k ≤ 4, k++,
     For[l = 1, l ≤ 4, l++, kset2bot[[j, k]] =
        kset2bot[[j, k]] - bb[[i]] cc[[l]]
          ((ksidjbot.gsigg[[i, j, k, l]].
              etachbot)[[1, 1]])];];];]
kset2bot = FullSimplify[kset2bot];
MatrixForm[kset2bot]
"kset2bot"
etks2bot = Array[0 &, {4, 4}];
For[i = 1, i ≤ 4, i++,
 For[j = 1, j ≤ 4, j++, For[k = 1, k ≤ 4, k++,
     For[l = 1, l ≤ 4, l++, etks2bot[[j, k]] =
        etks2bot[[j, k]] - cc[[i]] bb[[l]]
          ((etadjbot.gsigg[[i, j, k, l]].
              ksichbot)[[1, 1]])];];];]
etks2bot = FullSimplify[etks2bot];
MatrixForm[etks2bot]
```

```
"etks2bot"
bc27 = Array[0 &, {4, 4}];
For[j = 1, j ≤ 4, j++,
 For[k = 1, k ≤ 4, k++, bc27[[j, k]] =
     (-2 I bbr[[j]] ccr[[k]] + 2 I bbr[[k]] ccr[[
          j]]) (ksidjbot.etachbot)[[1, 1]];];]
bceps = -2 (ksidjbot.etachbot)[[1, 1]]
  FullSimplify[Activate[TensorContract[
     Inactive[TensorProduct][bb, cc,
      LeviCivitaTensor[4]], {{1, 3}, {2, 6}}]]];
MatrixForm[bceps]
"bceps"
kset3bot =
  -2 FullSimplify[Activate[TensorContract[
       Inactive[TensorProduct][bb, gg, cc,
        btbot], {{1, 2}, {4, 6}}]]] + 2 Transpose[
      FullSimplify[Activate[TensorContract[
        Inactive[TensorProduct][bb, gg,
         cc, btbot], {{1, 2}, {4, 6}}]]]] +
   2 FullSimplify[Activate[TensorContract[
       Inactive[TensorProduct][bb, cc, gg,
        btbot], {{2, 3}, {1, 5}}]]] - 2 Transpose[
      FullSimplify[Activate[TensorContract[
        Inactive[TensorProduct][bb, cc,
         gg, btbot], {{2, 3}, {1, 5}}]]]] -
```

```
    2 FullSimplify[Activate[TensorContract[
        Inactive[TensorProduct][bb, cc, gg],
        {{1, 4}, {2, 3}}]]] btbot ;
MatrixForm[FullSimplify[kset2bot +
    etks2bot - bc27 - kset3bot - bceps]]
"kset2bot+etks2bot-bc-kset3bot-bceps"
etet2bot = Array[0 &, {4, 4}];
For[i = 1, i ≤ 4, i++,
 For[j = 1, j ≤ 4, j++, For[k = 1, k ≤ 4, k++,
     For[l = 1, l ≤ 4, l++, etet2bot[[j, k]] =
        etet2bot[[j, k]] - cc[[i]] cc[[l]]
          ((etadjbot.gsigg[[i, j, k, l]].
              etachbot)[[1, 1]])];];];]
etet2bot = FullSimplify[etet2bot];
MatrixForm[etet2bot]
"etet2bot"
etet3bot =
   -2 FullSimplify[Activate[TensorContract[
        Inactive[TensorProduct][cc, gg, cc,
         ctbot], {{1, 2}, {4, 6}}]]] + 2 Transpose[
      FullSimplify[Activate[TensorContract[
         Inactive[TensorProduct][cc, gg,
          cc, ctbot], {{1, 2}, {4, 6}}]]]] -
    FullSimplify[Activate[TensorContract[
        Inactive[TensorProduct][cc, gg, cc],
```

```
        {{1, 2}, {3, 4}}]]] ctbot ;
MatrixForm[FullSimplify[etet2bot - etet3bot]]
"etet2bot-etet3bot"
```

This fragment of computation contains two similar parts corresponding to the cases where ξ and η are eigenvectors of γ^5 with eigenvalue $+1$ and -1.

sig[[i,j]] – $\sigma^{\mu\nu}$, gmt[[i]] – γ^μ, bb[[i]] – B_μ, cc[[i]] – C_μ. The computation shows that $-B_\mu B_\lambda \bar{\xi}\gamma^\mu\sigma^{\nu\sigma}\gamma^\lambda\xi^c = -2B^\nu B_\lambda u^{\sigma\lambda} + 2B^\sigma B_\lambda u^{\nu\lambda} - B^\lambda B_\lambda u^{\nu\sigma}$,

$$-C_\mu B_\lambda \bar{\eta}\gamma^\mu\sigma^{\nu\sigma}\gamma^\lambda\xi^c - B_\mu C_\lambda \bar{\xi}\gamma^\mu\sigma^{\nu\sigma}\gamma^\lambda\eta^c = -2iB^\nu C^\sigma + 2iB^\sigma C^\nu - 2B^\nu C_\lambda v^{\sigma\lambda} + 2B^\sigma C_\lambda v^{\nu\lambda} - 2B^\lambda C_\lambda v^{\nu\sigma} + 2B_\mu C^\nu v^{\mu\sigma} - 2B_\mu C^\sigma v^{\mu\nu} \pm 2B_\mu C_\lambda \epsilon^{\mu\nu\sigma\lambda},$$

and $-C_\mu C_\lambda \bar{\eta}\gamma^\mu\sigma^{\nu\sigma}\gamma^\lambda\eta^c = -2C^\nu C_\lambda w^{\sigma\lambda} + 2C^\sigma C_\lambda w^{\nu\lambda} - C^\lambda C_\lambda w^{\nu\sigma}$.

Out[988]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[989]= zer

Out[1003]= {{{0, 0, -1, 0}, {0, 0, 0, -1}, {-1, 0, 0, 0}, {0, -1, 0, 0}}, {{0, 0, 0, 1}, {0, 0, 1, 0}, {0, -1, 0, 0}, {-1, 0, 0, 0}}, {{0, 0, 0, -i}, {0, 0, i, 0}, {0, i, 0, 0}, {-i, 0, 0, 0}}, {{0, 0, 1, 0}, {0, 0, 0, -1}, {-1, 0, 0, 0}, {0, 1, 0, 0}}}

Out[1004]= gmt

Out[1013]//MatrixForm=

$$\left(\begin{array}{l} -i \left((bb0 + bb1 - i\, bb2 + bb3)\ (bb0 - bb1 + i\, bb2 + bb3) \right. \\ \quad - (bb0 - i\, bb1 - bb2 + bb3)\ (bb0 + i\, bb1 + bb2 + bb3) \\ \qquad\qquad\qquad 2\, i\ ((bb0 + bb3)\ \mathrm{Conjugate} \end{array} \right.$$

Out[1014]= ksks2top

Out[1016]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[1017]= ksks2top−ksks3top

Out[1020]//MatrixForm=

```
-ⅈ (Conjugate[eta2] (((bb0 + bb3) (cc1 + ⅈ cc2) - (b
    -Conjugate[eta2] (((bb0 + bb3) (cc1 + ⅈ cc2) + (l
  ⅈ (Conjugate[eta2] (((bb1 - ⅈ bb2) (cc1 + ⅈ cc2) +
```

Out[1021]= kset2top

Out[1024]//MatrixForm=

```
-ⅈ (Conjugate[eta2] (((bb0 + bb3) (cc1 + ⅈ cc2) - (b
    -Conjugate[eta2] (((bb0 + bb3) (cc1 + ⅈ cc2) + (l
  ⅈ (Conjugate[eta2] (((bb1 - ⅈ bb2) (cc1 + ⅈ cc2) +
```

Out[1025]= etks2top

Out[1028]//MatrixForm=

```
                                                   0
 2 (bb3 cc2 - bb2 cc3) (Conjugate[eta2] Conjugate[ks
2 (-bb3 cc1 + bb1 cc3) (Conjugate[eta2] Conjugate[k
 2 (bb2 cc1 - bb1 cc2) (Conjugate[eta2] Conjugate[ks
```

Out[1029]= bceps

Out[1031]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[1032]= kset2top+etks2top−bc−kset3top−bceps

Out[1035]//MatrixForm=

$$\begin{pmatrix} -i\ \big((cc0 + cc1 - i\ cc2 + cc3)\ (cc0 - cc1 + i\ cc2 + cc3) \\ -(cc0 - i\ cc1 - cc2 + cc3)\ (cc0 + i\ cc1 + cc2 + cc3) \\ 2\ i\ ((cc0 + cc3)\ \text{Conjugate} \end{pmatrix}$$

Out[1036]= etet2top

Out[1038]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[1039]= etet2top−etet3top

Out[1042]//MatrixForm=

$$\begin{pmatrix} -i\ \big(-(bb1 - i\ bb2)^2 + (bb0 - bb3)^2\big)\ \text{Conjugate}[ksi3]^2 \\ -\big((bb1 - i\ bb2)^2 + (bb0 - bb3)^2\big)\ \text{Conjugate}[ksi3]^2 \\ 2\ i\ ((bb0 - bb3)\ \text{Conjugate}[\end{pmatrix}$$

Out[1043]= ksks2bot

Out[1045]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[1046]= ksks2bot−ksks3bot

Out[1049]//MatrixForm=

```
 -ⅈ (Conjugate[eta3] ((-(bb1 - ⅈ bb2) (cc1 - ⅈ cc2) +
    Conjugate[eta3] ((-(bb1 - ⅈ bb2) (cc1 - ⅈ cc2) -
  ⅈ (Conjugate[eta3] ((-(bb0 - bb3) (cc1 - ⅈ cc2) - (
```

Out[1050]= kset2bot

Out[1053]//MatrixForm=

```
 -ⅈ (Conjugate[eta3] ((-(bb1 - ⅈ bb2) (cc1 - ⅈ cc2) +
    Conjugate[eta3] ((-(bb1 - ⅈ bb2) (cc1 - ⅈ cc2) -
  ⅈ (Conjugate[eta3] ((-(bb0 - bb3) (cc1 - ⅈ cc2) - (
```

Out[1054]= etks2bot

Out[1057]//MatrixForm=

```
                                             0
  -2 (bb3 cc2 - bb2 cc3) (-Conjugate[eta4] Conjugate
 -2 (-bb3 cc1 + bb1 cc3) (-Conjugate[eta4] Conjugate
  -2 (bb2 cc1 - bb1 cc2) (-Conjugate[eta4] Conjugate
```

Out[1058]= bceps

Out[1060]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[1061]= kset2bot+etks2bot-bc-kset3bot-bceps

Out[1064]//MatrixForm=

```
 -ⅈ (-(cc1 - ⅈ cc2)^2 + (cc0 - cc3)^2) Conjugate[eta3]^2
    -((cc1 - ⅈ cc2)^2 + (cc0 - cc3)^2) Conjugate[eta3]^2
                          2 ⅈ ((cc0 - cc3) Conjugate[e
```

Out[1065]= etet2bot

Out[1067]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[1068]= etet2bot−etet3bot

This fragment of computation contains two similar parts corresponding to the cases where ξ and η are eigenvectors of γ^5 with eigenvalue +1 and −1.

sig[[i,j]] – $\sigma^{\mu\nu}$, gmt[[i]] – γ^μ, bb[[i]] – B_μ, cc[[i]] – C_μ. The computation shows that $-B_\mu B_\lambda \bar{\xi}\gamma^\mu\sigma^{\nu\sigma}\gamma^\lambda\xi^c = -2B^\nu B_\lambda u^{\sigma\lambda} + 2B^\sigma B_\lambda u^{\nu\lambda} - B^\lambda B_\lambda u^{\nu\sigma}$,

Out[1069]=

$$-C_\mu B_\lambda \bar{\eta}\gamma^\mu\sigma^{\nu\sigma}\gamma^\lambda\xi^c - B_\mu C_\lambda \bar{\xi}\gamma^\mu\sigma^{\nu\sigma}\gamma^\lambda\eta^c = -2iB^\nu C^\sigma + 2iB^\sigma C^\nu - 2B^\nu C_\lambda v^{\sigma\lambda} + 2B^\sigma C_\lambda v^{\nu\lambda} - 2B^\lambda C_\lambda v^{\nu\sigma} + 2B_\mu C^\nu v^{\mu\sigma} - 2B_\mu C^\sigma v^{\mu\nu} \pm 2B_\mu C_\lambda \epsilon^{\mu\nu\sigma\lambda},$$

and $-C_\mu C_\lambda \bar{\eta}\gamma^\mu\sigma^{\nu\sigma}\gamma^\lambda\eta^c = -2C^\nu C_\lambda w^{\sigma\lambda} + 2C^\sigma C_\lambda w^{\nu\lambda} - C^\lambda C_\lambda w^{\nu\sigma}$.

In[1070]:=
```
"top calc"
uu = {{0, uu1, uu2, uu3}, {-uu1, 0, I uu3, -I uu2},
   {-uu2, -I uu3, 0, I uu1},
   {-uu3, I uu2, -I uu1, 0}};
MatrixForm[uu]
"uu"
uu25 = FullSimplify[Activate[TensorContract[
      Inactive[TensorProduct][uu, uu, gg, gg],
      {{1, 6}, {2, 8}, {3, 5}, {4, 7}}]]];
MatrixForm[uu25]
"uu25"
uuhodge = FullSimplify[(1 / 2) Activate[
      TensorContract[Inactive[TensorProduct][
```

```
        uu, LeviCivitaTensor[4], gg, gg],
       {{1, 7}, {2, 9}, {5, 8}, {6, 10}}]]];
MatrixForm[uuhodge]
"uuhodge"
FullSimplify[uu + I uuhodge]
vv = {{0, vv1, vv2, vv3}, {-vv1, 0, I vv3, -I vv2},
    {-vv2, -I vv3, 0, I vv1},
    {-vv3, I vv2, -I vv1, 0}};
uuvv25 = FullSimplify[Activate[TensorContract[
      Inactive[TensorProduct][uu, vv, gg, gg],
      {{1, 6}, {2, 8}, {3, 5}, {4, 7}}]]];
MatrixForm[uuvv25]
"uuvv25"
MatrixForm[gg];
uuvv = FullSimplify[Activate[
     TensorContract[Inactive[TensorProduct][
       uu, gg, vv], {{2, 3}, {4, 5}}]]];
MatrixForm[uuvv]
"uuvv"
uuvvc = FullSimplify[
   uuvv /. {vv1 → (-uu2 vv2 - uu3 vv3) / uu1}];
MatrixForm[uuvvc]
"uuvvc"
uuvvc2 =
  FullSimplify[uuvvc /. {uu1^2 + uu2^2 → -uu3^2,
```

```
    uu1^2 + uu3^2 → -uu2^2,
    uu2^2 + uu3^2 → -uu1^2}];
MatrixForm[uuvvc2]
"uuvvc2"
uuvvc3 =
 FullSimplify[uuvvc2 uu1 / (uu3 vv2 - uu2 vv3)];
MatrixForm[uuvvc3]
"uuvvc3"
MatrixForm[uuvvc3 + I uu]
MatrixForm[
 uuvv + I uu /. {uu1 vv1 + uu2 vv2 + uu3 vv3 → 0,
   -uu1 vv1 - uu2 vv2 - uu3 vv3 → 0}]
"uuvv+I uu/.{uu1 vv1+uu2 vv2+uu3 vv3→0}"
{uu1, uu2, uu3} +
 Cross[{uu1, uu2, uu3}, {vv1, vv2, vv3}]
"{uu1,uu2,uu3}+Cross[{uu1,uu2,uu3},{vv1,vv2,vv3
  }]"
"bottom calc"
uu = {{0, uu1, uu2, uu3}, {-uu1, 0, -I uu3, I uu2},
  {-uu2, I uu3, 0, -I uu1},
  {-uu3, -I uu2, I uu1, 0}};
MatrixForm[uu]
"uu"
uu25 = FullSimplify[Activate[TensorContract[
     Inactive[TensorProduct][uu, uu, gg, gg],
```

```
      {{1, 6}, {2, 8}, {3, 5}, {4, 7}}]]];
MatrixForm[uu25]
"uu25"
uuhodge = FullSimplify[(1 / 2) Activate[
      TensorContract[Inactive[TensorProduct][
        uu, LeviCivitaTensor[4], gg, gg],
       {{1, 7}, {2, 9}, {5, 8}, {6, 10}}]]];
MatrixForm[uuhodge]
"uuhodge"
FullSimplify[uu - I uuhodge]
vv = {{0, vv1, vv2, vv3}, {-vv1, 0, -I vv3, I vv2},
   {-vv2, I vv3, 0, -I vv1},
   {-vv3, -I vv2, I vv1, 0}};
MatrixForm[vv]
"vv"
uuvv25 = FullSimplify[Activate[TensorContract[
      Inactive[TensorProduct][uu, vv, gg, gg],
      {{1, 6}, {2, 8}, {3, 5}, {4, 7}}]]];
MatrixForm[uuvv25]
"uuvv25"
MatrixForm[gg];
uuvv = FullSimplify[Activate[
     TensorContract[Inactive[TensorProduct][
       uu, gg, vv], {{2, 3}, {4, 5}}]]]];
MatrixForm[uuvv]
```

```
"uuvv"
uuvvc = FullSimplify[
  uuvv /. {vv1 → (-uu2 vv2 - uu3 vv3) / uu1}];
MatrixForm[uuvvc]
"uuvvc"
uuvvc2 =
 FullSimplify[uuvvc /. {uu1^2 + uu2^2 → -uu3^2,
    uu1^2 + uu3^2 → -uu2^2,
    uu2^2 + uu3^2 → -uu1^2}];
MatrixForm[uuvvc2]
"uuvvc2"
uuvvc3 =
 FullSimplify[uuvvc2 (-uu1) / (uu3 vv2 - uu2 vv3)];
MatrixForm[uuvvc3]
"uuvvc3"
MatrixForm[uuvvc3 + I uu]
MatrixForm[
 uuvv + I uu /. {uu1 vv1 + uu2 vv2 + uu3 vv3 → 0,
   -uu1 vv1 - uu2 vv2 - uu3 vv3 → 0}]
"uuvv+I uu/.{uu1 vv1+uu2 vv2+uu3 vv3→0}"
{uu1, uu2, uu3} -
 Cross[{uu1, uu2, uu3}, {vv1, vv2, vv3}]
"{uu1,uu2,uu3}-
  Cross[{uu1,uu2,uu3},{vv1,vv2,vv3}]"
```

This fragment of computation contains two similar parts corresponding to the cases where the upper or lower signs are chosen in the section "Formulation in terms of antisymmetric second-rank tensors".

uu[[i,j]] – $u^{\mu\nu}$, vv[[i,j]] – $v^{\mu\nu}$, uuvv[[i,j]] – $u^{\mu}{}_{\sigma}v^{\sigma\nu}$, uu25 – $u^{\mu\nu}u_{\mu\nu}$, uuvv25 – $v^{\mu\nu}u_{\mu\nu}$, uuhodge[[i,j]] – $\star u^{\mu\nu}$.

uuvvc – uuvv after replacement $v^{01} \to (-u^{02}v^{02} - u^{03}v^{03})/u^{01}$.

uuvvc2 – uuvvc after replacement $(u^{01})^2 + (u^{02})^2 \to -(u^{03})^2, (u^{01})^2 + (u^{03})^2 \to -(u^{02})^2, (u^{02})^2 + (u^{03})^2 \to -(u^{01})^2$.

uuvvc3 – uuvvc2 after multiplication by $\pm\frac{u^{01}}{u^{03}v^{02} - u^{02}v^{03}}$ (we are trying to prove that if $\pm u^{01} = u^{03}v^{02} - u^{02}v^{03}, u^{\mu\nu}u_{\mu\nu} = 0, v^{\mu\nu}u_{\mu\nu} = 0$, then $u^{\mu}{}_{\sigma}v^{\sigma\nu} = -iu^{\mu\nu}$).

The computation shows that $v^{\mu\nu}u_{\mu\nu} = -4(v^{01}u^{01} + v^{02}u^{02} + v^{03}u^{03})$, the equality $u^{\mu}{}_{\sigma}v^{\sigma\nu} = -iu^{\mu\nu}$ is generally (if $u^{01} \neq 0$) equivalent to $\pm u^{01} = u^{03}v^{02} - u^{02}v^{03}$, and $u^{\mu}{}_{\sigma}v^{\sigma\nu} = -iu^{\mu\nu}$ implies $\boldsymbol{u} = \mp\boldsymbol{u} \times \boldsymbol{v}$, where, for example, $\boldsymbol{u} = (u^{01}, u^{02}, u^{03})$.

Out[1070]= top calc

Out[1071]//MatrixForm=

$$\begin{pmatrix} 0 & uu1 & uu2 & uu3 \\ -uu1 & 0 & i\,uu3 & -i\,uu2 \\ -uu2 & -i\,uu3 & 0 & i\,uu1 \\ -uu3 & i\,uu2 & -i\,uu1 & 0 \end{pmatrix}$$

Out[1072]= uu

Out[1074]//MatrixForm=

$$-4\left(uu1^2 + uu2^2 + uu3^2\right)$$

Out[1075]= uu25

Out[1077]//MatrixForm=

$$\begin{pmatrix} 0 & i\,uu1 & i\,uu2 & i\,uu3 \\ -i\,uu1 & 0 & -uu3 & uu2 \\ -i\,uu2 & uu3 & 0 & -uu1 \\ -i\,uu3 & -uu2 & uu1 & 0 \end{pmatrix}$$

Out[1078]= uuhodge

Out[1079]= {{0, 0, 0, 0}, {0, 0, 0, 0},
{0, 0, 0, 0}, {0, 0, 0, 0}}

Out[1082]//MatrixForm=

$$-4\,(uu1\,vv1 + uu2\,vv2 + uu3\,vv3)$$

Out[1083]= uuvv25

Out[1086]//MatrixForm=

$$\begin{pmatrix} uu1\,vv1 + uu2\,vv2 + uu3\,vv3 & i\,(-uu3\,vv2 + uu2\,vv3) \\ i\,(uu3\,vv2 - uu2\,vv3) & -uu1\,vv1 - uu2\,vv2 - uu3\,vv \\ i\,(-uu3\,vv1 + uu1\,vv3) & -uu2\,vv1 + uu1\,vv2 \\ i\,(uu2\,vv1 - uu1\,vv2) & -uu3\,vv1 + uu1\,vv3 \end{pmatrix}$$

Out[1087]= uuvv

Out[1088]//MatrixForm=

$$\begin{pmatrix} 0 & i\,(-uu3\,vv2 + uu2\,vv3) & -\frac{i}{\ } \\ i\,(uu3\,vv2 - uu2\,vv3) & 0 & - \\ \frac{i\,(uu2\,uu3\,vv2 + (uu1^2 + uu3^2)\,vv3)}{uu1} & \frac{(uu1^2 + uu2^2)\,vv2 + uu2\,uu3\,vv3}{uu1} & \\ -\frac{i\,((uu1^2 + uu2^2)\,vv2 + uu2\,uu3\,vv3)}{uu1} & \frac{uu2\,uu3\,vv2 + (uu1^2 + uu3^2)\,vv3}{uu1} & \end{pmatrix}$$

Out[1089]= uuvvc

Out[1090]//MatrixForm=

$$\begin{pmatrix} 0 & i\,(-uu3\,vv2 + uu2\,vv3) & \frac{i\,uu2\,(-uu}{\ } \\ i\,(uu3\,vv2 - uu2\,vv3) & 0 & \frac{uu3\,(uu3}{\ } \\ \frac{i\,uu2\,(uu3\,vv2 - uu2\,vv3)}{uu1} & \frac{uu3\,(-uu3\,vv2 + uu2\,vv3)}{uu1} & \\ \frac{i\,uu3\,(uu3\,vv2 - uu2\,vv3)}{uu1} & \frac{uu2\,(uu3\,vv2 - uu2\,vv3)}{uu1} & -uu3\,vv \end{pmatrix}$$

Out[1091]= uuvvc2

Out[1092]//MatrixForm=

$$\begin{pmatrix} 0 & -i\,uu1 & -i\,uu2 & -i\,uu3 \\ i\,uu1 & 0 & uu3 & -uu2 \\ i\,uu2 & -uu3 & 0 & uu1 \\ i\,uu3 & uu2 & -uu1 & 0 \end{pmatrix}$$

Out[1093]= uuvvc3

Out[1094]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[1095]//MatrixForm=

0	ⅈ uu1 + ⅈ (−uu3 vv2 + ı
−ⅈ uu1 + ⅈ (uu3 vv2 − uu2 vv3)	0
−ⅈ uu2 + ⅈ (−uu3 vv1 + uu1 vv3)	uu3 − uu2 vv1 + uu1
−ⅈ uu3 + ⅈ (uu2 vv1 − uu1 vv2)	−uu2 − uu3 vv1 + uu

Out[1096]= uuvv+I uu/.{uu1 vv1+uu2 vv2+uu3 vv3→0}

Out[1097]= {uu1 − uu3 vv2 + uu2 vv3,
uu2 + uu3 vv1 − uu1 vv3, uu3 − uu2 vv1 + uu1 vv2}

Out[1098]= {uu1,uu2,uu3}+Cross[{uu1,uu2,uu3},{vv1,vv2,vv3}
]

Out[1099]= bottom calc

Out[1100]//MatrixForm=

0	uu1	uu2	uu3
−uu1	0	−ⅈ uu3	ⅈ uu2
−uu2	ⅈ uu3	0	−ⅈ uu1
−uu3	−ⅈ uu2	ⅈ uu1	0

Out[1101]= uu

Out[1103]//MatrixForm=

$$-4\left(uu1^2 + uu2^2 + uu3^2\right)$$

Out[1104]= uu25

Out[1106]//MatrixForm=

0	−ⅈ uu1	−ⅈ uu2	−ⅈ uu3
ⅈ uu1	0	−uu3	uu2
ⅈ uu2	uu3	0	−uu1
ⅈ uu3	−uu2	uu1	0

Out[1107]= uuhodge

Out[1108]= {{0, 0, 0, 0}, {0, 0, 0, 0},
{0, 0, 0, 0}, {0, 0, 0, 0}}

Out[1109]//MatrixForm=

$$\begin{pmatrix} 0 & vv1 & vv2 & vv3 \\ -vv1 & 0 & -i\ vv3 & i\ vv2 \\ -vv2 & i\ vv3 & 0 & -i\ vv1 \\ -vv3 & -i\ vv2 & i\ vv1 & 0 \end{pmatrix}$$

Out[1110]= vv

Out[1112]//MatrixForm=

$$-4\ (uu1\ vv1 + uu2\ vv2 + uu3\ vv3)$$

Out[1113]= uuvv25

Out[1116]//MatrixForm=

$$\begin{pmatrix} uu1\ vv1 + uu2\ vv2 + uu3\ vv3 & i\ (uu3\ vv2 - uu2\ vv3) \\ i\ (-uu3\ vv2 + uu2\ vv3) & -uu1\ vv1 - uu2\ vv2 - uu3\ vv \\ i\ (uu3\ vv1 - uu1\ vv3) & -uu2\ vv1 + uu1\ vv2 \\ i\ (-uu2\ vv1 + uu1\ vv2) & -uu3\ vv1 + uu1\ vv3 \end{pmatrix}$$

Out[1117]= uuvv

Out[1118]//MatrixForm=

$$\begin{pmatrix} 0 & i\ (uu3\ vv2 - uu2\ vv3) & \frac{i\ (uu2}{} \\ i\ (-uu3\ vv2 + uu2\ vv3) & 0 & -\frac{(uu}{} \\ -\frac{i\ (uu2\ uu3\ vv2 + (uu1^2 + uu3^2)\ vv3)}{uu1} & \frac{(uu1^2 + uu2^2)\ vv2 + uu2\ uu3\ vv3}{uu1} & \\ \frac{i\ ((uu1^2 + uu2^2)\ vv2 + uu2\ uu3\ vv3)}{uu1} & \frac{uu2\ uu3\ vv2 + (uu1^2 + uu3^2)\ vv3}{uu1} & -u \end{pmatrix}$$

Out[1119]= uuvvc

Out[1120]//MatrixForm=

$$\begin{pmatrix} 0 & i\ (uu3\ vv2 - uu2\ vv3) & \frac{i\ uu2\ (uu3}{} \\ i\ (-uu3\ vv2 + uu2\ vv3) & 0 & \frac{uu3\ (uu3}{} \\ \frac{i\ uu2\ (-uu3\ vv2 + uu2\ vv3)}{uu1} & \frac{uu3\ (-uu3\ vv2 + uu2\ vv3)}{uu1} & \\ \frac{i\ uu3\ (-uu3\ vv2 + uu2\ vv3)}{uu1} & \frac{uu2\ (uu3\ vv2 - uu2\ vv3)}{uu1} & -uu3\ vv \end{pmatrix}$$

Out[1121]= uuvvc2

Out[1122]//MatrixForm=

$$\begin{pmatrix} 0 & -i\,uu1 & -i\,uu2 & -i\,uu3 \\ i\,uu1 & 0 & -uu3 & uu2 \\ i\,uu2 & uu3 & 0 & -uu1 \\ i\,uu3 & -uu2 & uu1 & 0 \end{pmatrix}$$

Out[1123]= uuvvc3

Out[1124]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[1125]//MatrixForm=

$$\begin{pmatrix} 0 & i\,uu1 + i\,(uu3\,vv2 - uu \\ -i\,uu1 + i\,(-uu3\,vv2 + uu2\,vv3) & 0 \\ -i\,uu2 + i\,(uu3\,vv1 - uu1\,vv3) & -uu3 - uu2\,vv1 + uu \\ -i\,uu3 + i\,(-uu2\,vv1 + uu1\,vv2) & uu2 - uu3\,vv1 + uu1 \end{pmatrix}$$

Out[1126]= uuvv+I uu/.{uu1 vv1+uu2 vv2+uu3 vv3→0}

Out[1127]= {uu1 + uu3 vv2 − uu2 vv3,
uu2 − uu3 vv1 + uu1 vv3, uu3 + uu2 vv1 − uu1 vv2}

Out[1128]= {uu1,uu2,uu3} −
Cross[{uu1,uu2,uu3},{vv1,vv2,vv3}]

This fragment of computation contains two similar parts corresponding to the cases where the upper or lower signs are chosen in the section "Formulation in terms of antisymmetric second-rank tensors".

uu[[i,j]] – $u^{\mu\nu}$, vv[[i,j]] – $v^{\mu\nu}$, uuvv[[i,j]] – $u^{\mu}{}_{\sigma}v^{\sigma\nu}$, uu25 – $u^{\mu\nu}u_{\mu\nu}$, uuvv25 – $v^{\mu\nu}u_{\mu\nu}$, uuhodge[[i,j]] – $\star u^{\mu\nu}$.

uuvvc – uuvv after replacement $v^{01} \to (-u^{02}v^{02} - u^{03}v^{02})/u^{01}$.

Out[1129]= uuvvc2 – uuvvc after replacement $(u^{01})^2 + (u^{02})^2 \to -(u^{03})^2, (u^{01})^2 + (u^{03})^2 \to -(u^{02})^2, (u^{02})^2 + (u^{03})^2 \to -(u^{01})^2$.

uuvvc3 – uuvvc2 after multiplication by $\pm\frac{u^{01}}{u^{03}v^{02}-u^{02}v^{03}}$ (we are trying to prove that if $\pm u^{01} = u^{03}v^{02} - u^{02}v^{03}, u^{\mu\nu}u_{\mu\nu} = 0, v^{\mu\nu}u_{\mu\nu} = 0$, then $u^{\mu}{}_{\sigma}v^{\sigma\nu} = -iu^{\mu\nu}$).

The computation shows that $v^{\mu\nu}u_{\mu\nu} = -4(v^{01}u^{01} + v^{02}u^{02} + v^{03}u^{03})$, the equality $u^{\mu}{}_{\sigma}v^{\sigma\nu} = -iu^{\mu\nu}$ is generally (if $u^{01} \neq 0$) equivalent to $\pm u^{01} = u^{03}v^{02} - u^{02}v^{03}$, and $u^{\mu}{}_{\sigma}v^{\sigma\nu} = -iu^{\mu\nu}$ implies $\boldsymbol{u} = \mp\boldsymbol{u} \times \boldsymbol{v}$, where, for example, $\boldsymbol{u} = (u^{01}, u^{02}, u^{03})$.

In[1130]:=
```
"top calc"
attopsp = attop /. {ksi1 → 1, ksi2 → 0};
MatrixForm[attopsp]
"attopsp"
attopsphodge =
  FullSimplify[(1 / 2) Activate[TensorContract[
      Inactive[TensorProduct][attopsp,
       LeviCivitaTensor[4], gg, gg],
      {{1, 7}, {2, 9}, {5, 8}, {6, 10}}]]];
MatrixForm[attopsphodge]
"attopsphodge"
FullSimplify[attopsp + I attopsphodge]
bttopsp =
 bttop /. {ksi1 → 1, ksi2 → 0, eta1 → 0, eta2 → 1};
MatrixForm[bttopsp]
"bttopsp"
```

```
cttopsp = cttop /. {eta1 → 0, eta2 → 1};
MatrixForm[cttopsp]
"cttopsp"
MatrixForm[ksidj.etach /.
  {ksi3 → 0, ksi4 → 0, eta3 → 0, eta4 → 0,
   ksi1 → 1, ksi2 → 0, eta1 → 0, eta2 → 1}]
"ksidj.etach/.{ksi3→0,ksi4→0,eta3→0,eta4→0,ksi1
  →1,ksi2→0,eta1→0,eta2→1}"
FullSimplify[Activate[TensorContract[
   Inactive[TensorProduct][attopsp, gg, gg,
    attopsp], {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]]
"attopsp^2"
FullSimplify[Activate[
  TensorContract[Inactive[TensorProduct][
    bttopsp, gg, gg, bttopsp],
   {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]]
"bttopsp^2"
FullSimplify[Activate[
  TensorContract[Inactive[TensorProduct][
    cttopsp, gg, gg, cttopsp],
   {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]]
"cttopsp^2"
FullSimplify[Activate[TensorContract[
    Inactive[TensorProduct][attopsp, gg,
     bttopsp], {{2, 3}, {4, 5}}]]] + I attopsp
```

```
"[attopsp,gg,bttopsp],{{2,3},{4,5}}+I attopsp]"
FullSimplify[Activate[
  TensorContract[Inactive[TensorProduct][
    bttopsp, gg, gg, cttopsp],
   {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]]
"bttopsp cttopsp"
FullSimplify[Activate[
  TensorContract[Inactive[TensorProduct][
    attopsp, gg, gg, cttopsp],
   {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]]
"attopsp cttopsp"
"bot calc"
atbotsp = atbot /. {ksi3 → -1, ksi4 → 0};
MatrixForm[atbotsp]
"atbotsp"
atbotsphodge =
  FullSimplify[(1 / 2) Activate[TensorContract[
      Inactive[TensorProduct][atbotsp,
       LeviCivitaTensor[4], gg, gg],
      {{1, 7}, {2, 9}, {5, 8}, {6, 10}}]]];
MatrixForm[atbotsphodge]
"atbotsphodge"
FullSimplify[atbotsp - I atbotsphodge]
btbotsp =
 btbot /. {ksi3 → -1, ksi4 → 0, eta3 → 0, eta4 → 1};
```

```
MatrixForm[btbotsp]
"btbotsp"
ctbotsp = ctbot /. {eta3 → 0, eta4 → 1};
MatrixForm[ctbotsp]
"ctbotsp"
MatrixForm[ksidj.etach /.
   {ksi3 → -1, ksi4 → 0, eta3 → 0, eta4 → 1,
    ksi1 -> 0, ksi2 -> 0, eta1 -> 0, eta2 -> 0}]
"ksidj.etach/.{ksi3→-1,ksi4→0,eta3→0,eta4→1,
  ksi1->0,ksi2->0,eta1->0,eta2->0}"
FullSimplify[Activate[
  TensorContract[Inactive[TensorProduct][
     atbotsp, gg, gg, atbotsp],
    {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]]
"atbotsp^2"
FullSimplify[Activate[
  TensorContract[Inactive[TensorProduct][
     btbotsp, gg, gg, btbotsp],
    {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]]
"btbotsp^2"
FullSimplify[Activate[
  TensorContract[Inactive[TensorProduct][
     ctbotsp, gg, gg, ctbotsp],
    {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]]
"ctbotsp^2"
```

```
FullSimplify[Activate[TensorContract[
    Inactive[TensorProduct][atbotsp, gg,
     btbotsp], {{2, 3}, {4, 5}}]]] + I atbotsp
"[atbotsp,gg,btbotsp],{{2,3},{4,5}}+I atbotsp]"
FullSimplify[Activate[
  TensorContract[Inactive[TensorProduct][
    btbotsp, gg, gg, ctbotsp],
   {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]]
"btbotsp ctbotsp"
FullSimplify[Activate[
  TensorContract[Inactive[TensorProduct][
    atbotsp, gg, gg, ctbotsp],
   {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]]
"atbotsp ctbotsp"
```

This fragment of computation contains two similar parts corresponding to the cases where the upper or lower signs are chosen in the section "Formulation in terms of antisymmetric second-rank tensors".

attopsp[[i,j]], bttopsp[[i,j]], cttopsp[[i,j]] – tensors $u^{\mu\nu}, v^{\mu\nu}, w^{\mu\nu}$ for upper signs after substitution of specific values of the spinors

$$\xi = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \eta = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

atbotsp[[i,j]], btbotsp[[i,j]], ctbotsp[[i,j]] – tensors $u^{\mu\nu}, v^{\mu\nu}, w^{\mu\nu}$ for lower signs after substitution of specific values of the spinors

$$\xi = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \eta = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The computation shows that several equalities hold for the specific values of $\xi, \eta, u^{\mu\nu}, v^{\mu\nu}, w^{\mu\nu}$.

Out[1130]= top calc

Out[1131]//MatrixForm=

$$\begin{pmatrix} 0 & i & 1 & 0 \\ -i & 0 & 0 & -i \\ -1 & 0 & 0 & -1 \\ 0 & i & 1 & 0 \end{pmatrix}$$

Out[1132]= attopsp

Out[1134]//MatrixForm=

$$\begin{pmatrix} 0 & -1 & i & 0 \\ 1 & 0 & 0 & 1 \\ -i & 0 & 0 & -i \\ 0 & -1 & i & 0 \end{pmatrix}$$

Out[1135]= attopsphodge

Out[1136]= {{0, 0, 0, 0}, {0, 0, 0, 0},
{0, 0, 0, 0}, {0, 0, 0, 0}}

Out[1137]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

Out[1138]= bttopsp

Out[1139]//MatrixForm=

$$\begin{pmatrix} 0 & -i & 1 & 0 \\ i & 0 & 0 & -i \\ -1 & 0 & 0 & 1 \\ 0 & i & -1 & 0 \end{pmatrix}$$

Out[1140]= cttopsp

Out[1141]//MatrixForm=

$$\begin{pmatrix} 1 \end{pmatrix}$$

Out[1142]= ksidj.etach/.{ksi3→0,ksi4→0,eta3→0,eta4→0,ksi1→1,ksi2→0,eta1→0,eta2→1}

Out[1143]= 0

Out[1144]= attopsp^2

Out[1145]= 4

Out[1146]= bttopsp^2

Out[1147]= 0

Out[1148]= cttopsp^2

Out[1149]= {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

Out[1150]= [attopsp,gg,bttopsp],{{2,3},{4,5}}+I attopsp]

Out[1151]= 0

Out[1152]= bttopsp cttopsp

Out[1153]= −8

Out[1154]= attopsp cttopsp

Out[1155]= bot calc

Out[1156]//MatrixForm=

$$\begin{pmatrix} 0 & i & 1 & 0 \\ -i & 0 & 0 & i \\ -1 & 0 & 0 & 1 \\ 0 & -i & -1 & 0 \end{pmatrix}$$

Out[1157]= atbotsp

Out[1159]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & -i & 0 \\ -1 & 0 & 0 & 1 \\ i & 0 & 0 & -i \\ 0 & -1 & i & 0 \end{pmatrix}$$

Out[1160]= atbotsphodge

Out[1161]= {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

Out[1162]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

Out[1163]= btbotsp

Out[1164]//MatrixForm=

$$\begin{pmatrix} 0 & -i & 1 & 0 \\ i & 0 & 0 & i \\ -1 & 0 & 0 & -1 \\ 0 & -i & 1 & 0 \end{pmatrix}$$

Out[1165]= ctbotsp

Out[1166]//MatrixForm=

$$\begin{pmatrix} 1 \end{pmatrix}$$

Out[1167]= ksidj.etach/.{ksi3→-1,ksi4→0,eta3→0,eta4→1,ksi1->0,ksi2->0,eta1->0,eta2->0}

Out[1168]= 0

Out[1169]= atbotsp^2

Out[1170]= 4

Out[1171]= btbotsp^2

Out[1172]= 0

Out[1173]= ctbotsp^2

Out[1174]= {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

Out[1175]= [atbotsp,gg,btbotsp],{{2,3},{4,5}}+I atbotsp]

Out[1176]= 0

Out[1177]= btbotsp ctbotsp

Out[1178]= −8

Out[1179]= atbotsp ctbotsp

This fragment of computation contains two similar parts corresponding to the cases where the upper or lower signs are chosen in the section "Formulation in terms of antisymmetric second-rank tensors".

attopsp[[i,j]], bttopsp[[i,j]], cttopsp[[i,j]] – tensors $u^{\mu\nu}, v^{\mu\nu}, w^{\mu\nu}$ for upper signs after substitution of specific values of the spinors

$$\xi = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \eta = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Out[1180]=

atbotsp[[i,j]], btbotsp[[i,j]], ctbotsp[[i,j]] – tensors $u^{\mu\nu}, v^{\mu\nu}, w^{\mu\nu}$ for lower signs after substitution of specific values of the spinors

$$\xi = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \eta = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The computation shows that several equalities hold for the specific values of $\xi, \eta, u^{\mu\nu}, v^{\mu\nu}, w^{\mu\nu}$.

In[1181]:=
```
"top calc"
uu = {{0, uu1, uu2, uu3}, {-uu1, 0, I uu3, -I uu2},
   {-uu2, -I uu3, 0, I uu1},
   {-uu3, I uu2, -I uu1, 0}};
MatrixForm[uu]
"uu"
ff = {{0, -ee1, -ee2, -ee3}, {ee1, 0, -hh3, hh2},
   {ee2, hh3, 0, -hh1}, {ee3, -hh2, hh1, 0}};
```

```
MatrixForm[ff]
"ff"
ffuu = FullSimplify[Activate[TensorContract[
     Inactive[TensorProduct][ff, gg, gg, uu],
     {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]];
MatrixForm[ffuu]
"ffuu"
"bottom calc"
uu = {{0, uu1, uu2, uu3}, {-uu1, 0, -I uu3, I uu2},
   {-uu2, I uu3, 0, -I uu1},
   {-uu3, -I uu2, I uu1, 0}};
MatrixForm[uu]
"uu"
ff = {{0, -ee1, -ee2, -ee3}, {ee1, 0, -hh3, hh2},
   {ee2, hh3, 0, -hh1}, {ee3, -hh2, hh1, 0}};
MatrixForm[ff]
"ff"
ffuu = FullSimplify[Activate[TensorContract[
     Inactive[TensorProduct][ff, gg, gg, uu],
     {{1, 3}, {2, 5}, {4, 7}, {6, 8}}]]];
MatrixForm[ffuu]
```

This fragment of computation contains two similar parts corresponding to the cases where the upper or lower signs are chosen in the section "Formulation in terms of 3D vectors".
uu[[i,j]] – $u^{\rho\nu}$, ee1, ee2, ee3 – E^1, E^2, E^3, hh1, hh2, hh3 – H^1, H^2, H^3, ff[[i,j]] – $F^{\mu\nu}$.
The computation shows that, e. g., $F_{\rho\nu}u^{\rho\nu} = 2(\boldsymbol{u}\cdot(\boldsymbol{E}\mp i\boldsymbol{H}))$.

Out[1181]= top calc

Out[1182]//MatrixForm=

$$\begin{pmatrix} 0 & uu1 & uu2 & uu3 \\ -uu1 & 0 & i\, uu3 & -i\, uu2 \\ -uu2 & -i\, uu3 & 0 & i\, uu1 \\ -uu3 & i\, uu2 & -i\, uu1 & 0 \end{pmatrix}$$

Out[1183]= uu

Out[1184]//MatrixForm=

$$\begin{pmatrix} 0 & -ee1 & -ee2 & -ee3 \\ ee1 & 0 & -hh3 & hh2 \\ ee2 & hh3 & 0 & -hh1 \\ ee3 & -hh2 & hh1 & 0 \end{pmatrix}$$

Out[1185]= ff

Out[1186]//MatrixForm=

$$2\, ee1\, uu1 - 2\, i\, (hh1\, uu1 + i\, ee2\, uu2 + hh2\, uu2 + i\, ee3\, uu3 + hh3\, uu3)$$

Out[1187]= ffuu

Out[1188]= bottom calc

Out[1189]//MatrixForm=

$$\begin{pmatrix} 0 & uu1 & uu2 & uu3 \\ -uu1 & 0 & -i\, uu3 & i\, uu2 \\ -uu2 & i\, uu3 & 0 & -i\, uu1 \\ -uu3 & -i\, uu2 & i\, uu1 & 0 \end{pmatrix}$$

Out[1190]= uu

Out[1191]//MatrixForm=

$$\begin{pmatrix} 0 & -ee1 & -ee2 & -ee3 \\ ee1 & 0 & -hh3 & hh2 \\ ee2 & hh3 & 0 & -hh1 \\ ee3 & -hh2 & hh1 & 0 \end{pmatrix}$$

Out[1192]= ff

Out[1193]//MatrixForm=

$$2\,(\text{ee1 uu1} + i\,\text{hh1 uu1} + \text{ee2 uu2} + i\,\text{hh2 uu2} + \text{ee3 uu3} + i\,\text{hh3 uu3})$$

Out[1194]=

This fragment of computation contains two similar parts corresponding to the cases where the upper or lower signs are chosen in the section "Formulation in terms of 3D vectors". uu[[i,j]] – $u^{\mu\nu}$, ee1, ee2, ee3 – E^1, E^2, E^3, hh1, hh2, hh3 – H^1, H^2, H^3, ff[[i,j]] – $F^{\mu\nu}$.

The computation shows that, e. g., $F_{\mu\nu}u^{\mu\nu} = 2(\boldsymbol{u}\cdot(\boldsymbol{E}\mp i\boldsymbol{H}))$.